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Originality and Importance of Technical Papers

THE SPECTRUM of originality in technical writing is extremely broad. It includes the revolutionary exposition of a new idea as well as the relatively simple extension of a well-established theory. Although papers revealing broad, new concepts are most desired, they are rare, and at times their meaning may be obscure, their value unobvious. Papers of lesser stature, those that are arranged from the sifted wisdom of more profound works, are also difficult to evaluate. If tutorial, they may serve a useful purpose, but when their intent and derivation is not clear, they may border on plagiarism, an offense most detrimental to professional standards and a practice to be discouraged by all technical publications.

While the majority of technical papers are not of momentous importance, they usually possess professional integrity. They extend technical knowledge by re-assessing and extending the works of others, and many fundamental theories have evolved slowly from these small additions to knowledge. Frequently, applications and simple modifications of existing theories, so obvious after they are revealed, are extremely useful in practice. The Smith chart and the Nichols chart are examples of elementary coordinate transformations which add little to the fundamental theory in their respective fields, but they do contribute greatly to ease of analysis and over-all understanding of the theory and its applications. It is clear that the importance of a technical paper is not measured entirely by its originality nor by its contribution to fundamental theory. Indeed, many fundamental papers lose their importance because they are written tersely by theorists with little time for expository detail. Consequently, they may be ignored until a clearly written tutorial paper interprets the basic idea of the paper and illustrates its usefulness.

For these reasons, we do not subscribe to the frequently received suggestion that technical papers should not be published if they could have been created easily from established theories, because the ease with which an idea could have been created cannot be judged after it has been revealed, and because all extensions of our knowledge have been based on the revelations of others. Therefore, it is our intent to publish clearly written papers whether of fundamental or tutorial importance if it is believed that they will contribute in any way to the general theory and basic understanding of automatic control.—*The Editor.*

The Issue in Brief

Accuracy Requirements of Nonlinear Compensation for Backlash, Donald Schulkind

The detrimental effects of backlash are analyzed with the describing function, and both linear and nonlinear compensation are devised. The nonlinear compensator effectively matches the reciprocal of the backlash function, thus theoretically eliminating the backlash. A noteworthy feature of the paper is that it considers the effects of a mismatch between the backlash and its simulated function. It is shown that great matching accuracy is not required for a significant improvement in performance.

Specifications of the Linear Feedback Sensitivity Function, William A. Mazer

Although one of the main functions of feedback is to reduce the detrimental effects of component variations on system performance, this characteristic is seldom analyzed quantitatively as it can be in terms of the system sensitivity function. This paper discusses the sensitivity function on the basis of minimizing the mean square variation of system response. This results in a different performance criterion which provides control over the variations of both the characteristic and forced responses with system parameter changes.

Synthesis of Linear, Multivariable Feedback Control Systems, Isaac M. Horowitz

A synthesis procedure is developed in this paper to reduce the system response sensitivity, to reduce system response to disturbances, and to realize a set of transmission functions in a multivariable controlled process where there are n independent inputs and m outputs with $n>1$ and $m\leq n$. The design procedures are illustrated with detailed examples in which there are large plant parameter variations.

Automatic Control of Vector Quantities, Part III, Allen S. Lange

This is the last article discussing the analysis and design of control systems involving vector quantities. The first article[1] discussed space vectors, the second[2] involved velocity vectors, and this paper is largely devoted to forces and moments which produce space accelerations, important components of which are sometimes neglected in simplified analyses.

[1] A. S. Lange, "The control of vector quantities, part I," IRE Trans. on Automatic Control, vol. AC-4, pp. 21–30; May, 1959.

[2] A. S. Lange, "The control of vector quantities, part II," *ibid.*, vol. AC-5, pp. 8–57; January, 1960.

Statistical Evaluation of Digital-Analog Systems for Finite Operating Time, R. B. Northrup and G. W. Johnson

This paper, rather long in its development of a frequency-domain mathematical model for comparing the performance of a linear sampled-data channel to a linear continuous channel, uses many well-known techniques to determine an explicit algebraic definition of ensemble mean-square error after a finite operating time. This problem is important in inertial and Doppler navigation systems, and it is hoped that the detailed development, including the illustrative examples, will be of use to those interested in this field.

A Mathematical Representation of Hydraulic Servomechanisms, J. J. Rodden

A detailed, basic representation of a high performance hydraulic servo actuator is given in this paper. Simplification and simulation of the equations are discussed and the supply accumulator and pump are included in the analysis which has been applied in the design of missile flight control systems where the rate limit characteristic of the actuator is significant.

A General Method for Deriving the Describing Functions for a Certain Class of Nonlinearities, R. Sridhar

The describing function for two general types of nonlinearities are derived and over twenty other describing functions are obtained directly from them. The describing functions are classified and the defining equations are grouped in a convenient tabular form.

Soviet Literature on Control Systems, P. L. Simmons and H. A. Pappo

Although this relatively short bibliography cannot be referred to as an exhaustive list of Russian control literature, it does cite many of the well-known articles published in Russia during recent years, it includes many annotations, and it should be of interest to those who are not familiar with Russian work in the control field.

Correspondence

Interesting properties, characteristics, and classifications of adaptive and optimalizing control systems are suggested by Pieter Eykhoff, a note on a third-order linear system is presented by Preston R. Clement, and in "Improved Transient Response in a Servo System with Input Modifications," B. Chatterjee derives an alternative way of synthesizing a "Posicast" control system.

Accuracy Requirements of Nonlinear Compensation for Backlash*

DONALD SCHULKIND†

Summary—**Recent advances in the analysis of nonlinear control systems have given rise to a more logical and systematic approach to the elimination of the detrimental effects of inherent nonlinearities. The describing function technique is used in this paper to analyze two typical servos. The servos are shown to exhibit a stable limit cycle and are stabilized by linear and nonlinear techniques. The nonlinear technique involves the insertion of an additional nonlinear feedback element which feeds back a distorted signal opposite in phase to the original distorted feedback signal. In particular, a method is developed in this paper for determining the accuracy required in the construction of a practical nonlinear compensating element.**

Introduction

SATURATION in amplifiers, backlash in gear trains, and dead space in motors are typical inherent nonlinearities which must be considered in the design of any practical servo. Numerous papers have been written analyzing the effects of these nonlinear elements.[1,2] Once these effects have been analyzed and are shown to affect seriously the performance of the servo, some means of compensation must be employed. The compensation technique used in this paper involves the insertion of an additional nonlinear element to compensate for the existing nonlinear element.[3] This paper presents a technique for determining the degree of accuracy required in the construction of the nonlinear compensating element for any servo under consideration.

Discussion

Development of the Problem

The block diagram of the first servo under discussion is shown in Fig. 1. In order to analyze the effect of the nonlinear element whose describing function is N, the frequency-dependent portion is separated from the magnitude-dependent portion of the loop (Fig. 2). If

$$1 + GN = 0 \tag{1}$$

or

$$G = -1/N \tag{2}$$

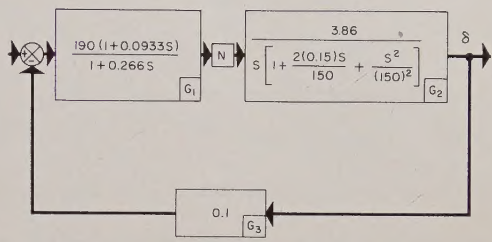

Fig. 1—Typical type-1 positioning servo for an air-to-surface missile.

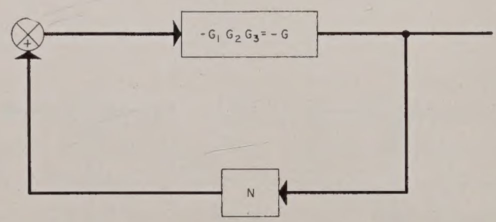

Fig. 2—Modification of Fig. 1.

is satisfied, then a limit cycle exists. When the curves representing G and $-1/N$ intersect, the condition for a limit cycle is satisfied. The manner in which these curves intersect determines whether or not the limit cycle is a stable one.

The Nyquist diagram for the frequency-dependent portion and the amplitude loci of the describing function for backlash (zero per cent) are shown in Fig. 3. The dotted curves and shaded area of Fig. 3 are explained in the section on nonlinear compensation of this loop. The frequency and magnitude of the limit cycle are determined in the following manner. From Fig. 3, the intersection of the frequency-dependent portion and the amplitude loci of the describing function (zero per cent) for backlash occurs at $\omega = 6$ rad/second and $|-1/|N||$ $= 8.0$ or $|N| = 0.125$. When $|N|$ is equal to 0.125, then $A/a = 1.09$ (see Appendix I) and the following equation results:

$$\begin{aligned}\delta_{osc} &= (A)(N)(G_2)(2) \\ &= (A)(|N|)\left[\frac{3.86}{S\left[1+\frac{2(0.15)S}{(150)}+\frac{S^2}{(150)^2}\right]}\right](2)\Bigg|_{S=j6} \\ &= 0.177(a) \text{ (peak-to-peak)}\end{aligned} \tag{3}$$

where a is the width of the backlash, A is the magnitude of the sinusoidal input to the nonlinear element, and

$$\omega_{osc} = 6 \text{ rad/second}. \tag{4}$$

* Manuscript received by the PGAC, July 21, 1959; revised manuscript received, December 23, 1959.

† Air Armament Div., Sperry Gyroscope Co., Div. of Sperry Rand Corp., Great Neck, N. Y.

[1] E. C. Johnson, "Sinusoidal analysis of feedback control systems containing nonlinear elements," *Trans. AIEE*, paper no. 52-154, vol. 71, pp. 169–181; April, 1952.

[2] N. B. Nichols, "Backlash in a velocity lag servomechanism," *Trans. AIEE*, pt. II, paper no. 53-394, vol. 72, pp. 462–467; January, 1953.

[3] G. Casserly and J. G. Truxal, "Measurement and stabilization of nonlinear feedback systems," 1956 IRE Convention Record, pt. 4, pp. 52–50.

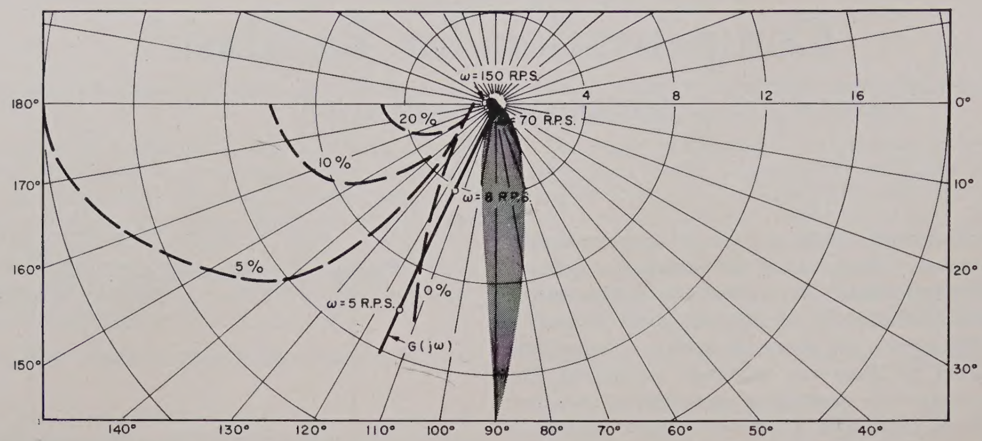

Fig. 3—Nyquist diagram for Fig. 2 with amplitude loci for nonlinear-compensated backlash.

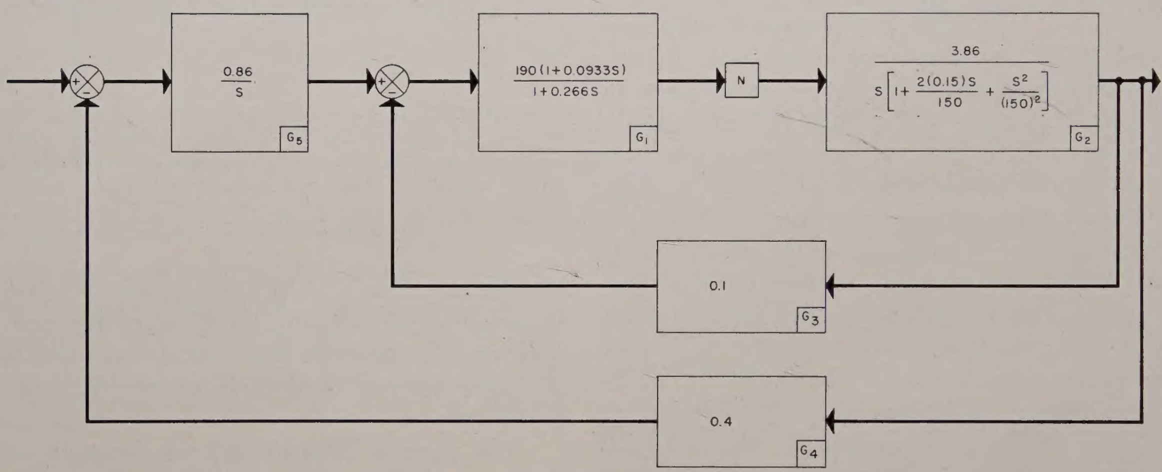

Fig. 4—Servo No. 2.

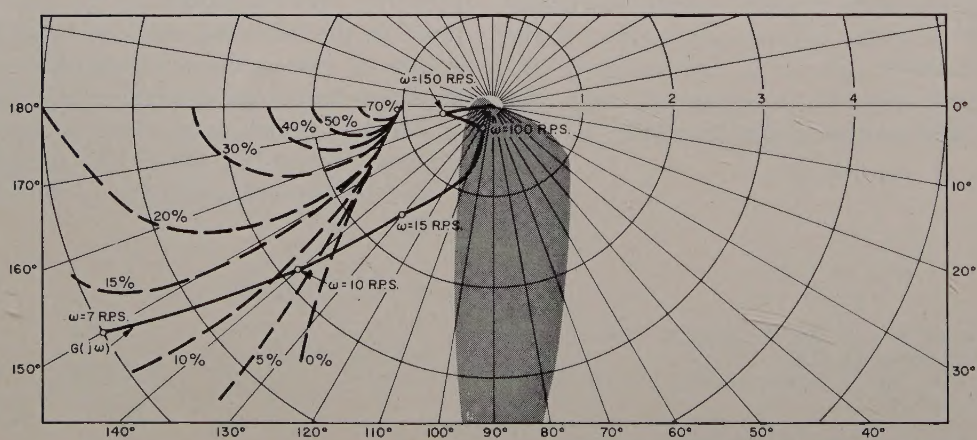

Fig. 5—Nyquist diagram for Fig. 4 with amplitude loci for nonlinear-compensated backlash.

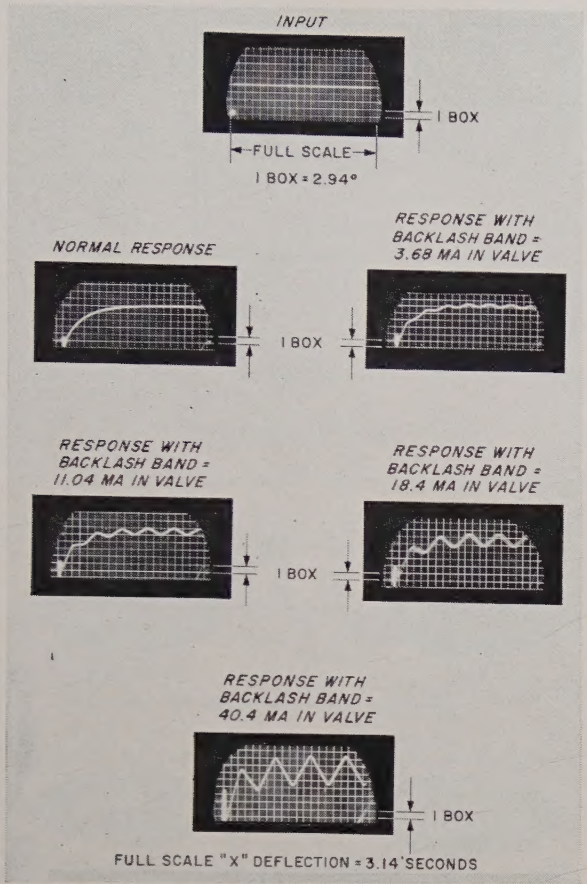

Fig. 6—High-speed analog computer results.

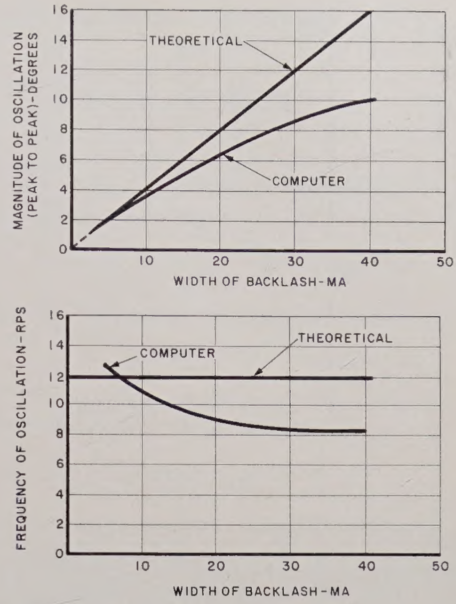

Fig. 7—Comparison of theoretical and computer results.

The block diagram of the second servo under discussion is shown in Fig. 4. The loop is analyzed in a manner similar to that for the first loop. The results are shown in Fig. 5. The frequency and magnitude of oscillation are:

$$\omega_{\text{osc}} = 12 \text{ rad/second} \tag{5}$$

$$\delta_{\text{osc}} = 0.402a \text{ (peak-to-peak)}. \tag{6}$$

The basic difference between the two loops is that the frequency-dependent portion of the loops as seen from the nonlinear element contains a pure $1/s$ in the first servo and a pure $1/s^2$ in the second servo. An analog computer study was performed with the results presented in Figs. 6 and 7.[4] A comparison of the results obtained from the analog computer study and theoretical computations shows that both agree favorably with the magnitude and frequency of oscillation noted on the physical equipment.

Elimination of the Limit Cycles

A study of the Nyquist diagram of Fig. 3 shows that additional phase lag introduced by the amplifier at low frequencies is sufficiently great to cause an intersection with the amplitude loci for backlash. If a lead-lag network[5] is inserted in the loop so that the over-all phase lag occurs at higher frequencies, no intersection occurs (Fig. 8). This technique was employed on the physical equipment with the results shown in Fig. 9.

If a nonlinear device having a describing function N' and a frequency-dependent portion G' is inserted into the loop, as shown in Fig. 10, the basic condition for a limit cycle is:

$$1 + NG_1G_2G_3 + N'G_1G' = 0. \tag{7}$$

For

$$G' = G_2G_3 = \frac{0.386}{S\left[1 + \frac{2(0.15)S}{150} + \frac{S^2}{(150)^2}\right]}$$

(perfect linear element), (7) may be written

$$\frac{-1}{N + N'} = G_1G_2G_3 = \left[\frac{190(1 + 0.0933S)}{1 + 0.266S}\right]$$

$$\cdot\left[\frac{3.86}{S\left[1 + \frac{2(0.15)S}{150} + \frac{S^2}{(150)^2}\right]}\right](0.1) \tag{8}$$

[4] It should be noted that backlash is given in terms of milliamperes (valve current) since the nonlinearity occurs between the input current to the torque motor which positions the hydraulic valve and the positioning of the hydraulic valve.

[5] A lead-lag network has a transfer function of the form

$$\frac{T_2}{T_1}\cdot\frac{1 + T_1S}{1 + T_2S} \quad \text{where} \quad T_1 > T_2.$$

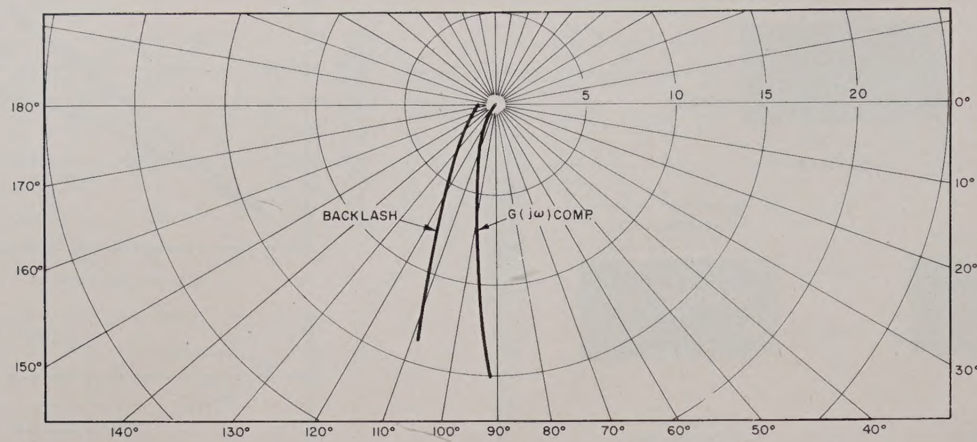

Fig. 8—Nyquist diagram for servo no. 1 with linear compensation and amplitude loci for backlash.

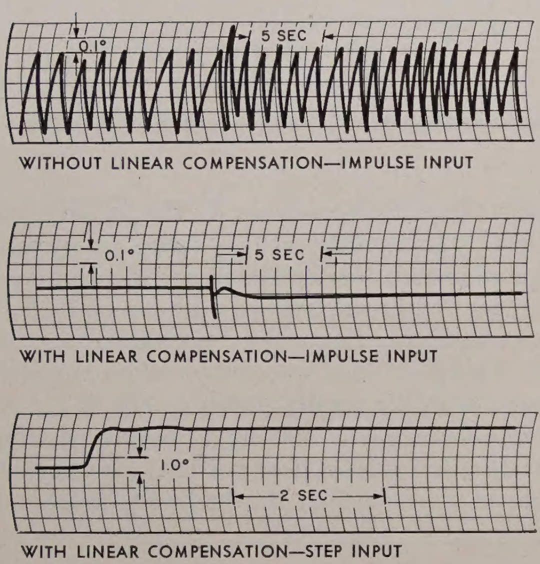

Fig. 9—Response of servo no. 1.

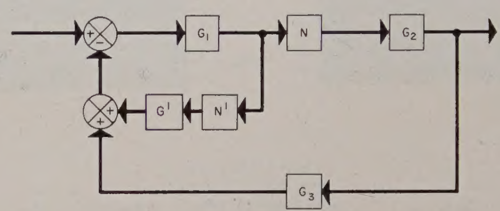

Fig. 10—Servo no. 1 with nonlinear compensation.

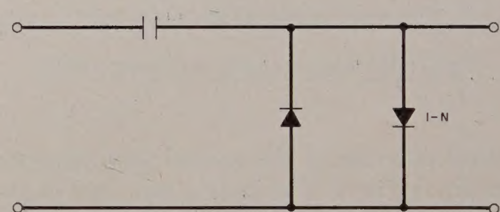

Fig. 11—Network simulation for $1-N$ where N is the describing function for backlash.

where the right-hand side is the Nyquist diagram of Fig. 3. The dashed curves of Fig. 3 represent a variation in N' where $N'=x(1-N)$ as x varies from 0 to 1.0 (0 per cent to 100 per cent). The figure 100 per cent represents perfect conpensation. It is seen from Fig. 3 that only 5 per cent or more nonlinear compensation is required for this servo to eliminate the limit cycle. A circuit developing a good approximation to $1-N$ is shown in Fig. 11. A discussion of the operation of this circuit is presented in Appendix II. For $N'=1-N$ (perfect nonlinear element), (7) may be written

$$\frac{-1}{N}=\frac{G_1G_2G_3-G_1G'}{1+G_1G'}. \tag{9}$$

The shaded area of Fig. 3 represents a variation in the linear element G' where $G'=K/(S+\alpha)$ as K varies from 0.193 to 0.772 and α varies from 0.193 to 0.772. It is seen from Fig. 3 that the elimination of the limit cycle is very insensitive to variations in the linear portion of the compensating device. The nonlinear element of Fig. 11 and a linear element $G'=0.386/(S+0.386)$ were incorporated into the physical equipment with the results shown in Fig. 12. An analog computer study was performed employing nonlinear compensation with the results shown in Fig. 13.

An examination of the Nyquist diagram for the second servo shows that pure differentiation is the only form of linear compensation that could eliminate the limit cycle, since the frequency-dependent portion of the loop has a high gain and $-180°$ phase shift at low frequencies and must approach zero gain at high frequencies. However, if pure differentiation is used, the positional accuracy of the servo is seriously affected. Therefore, only nonlinear compensation is considered for this servo.

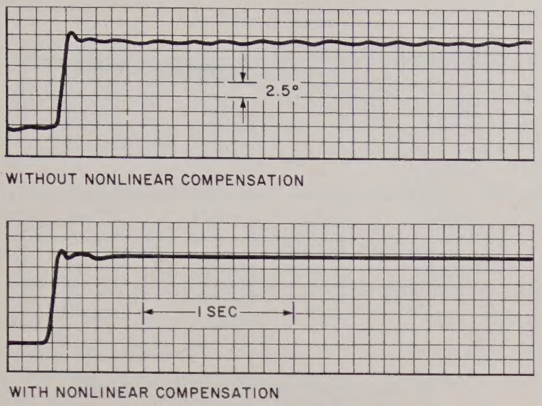

Fig. 12—Step response of servo no. 1.

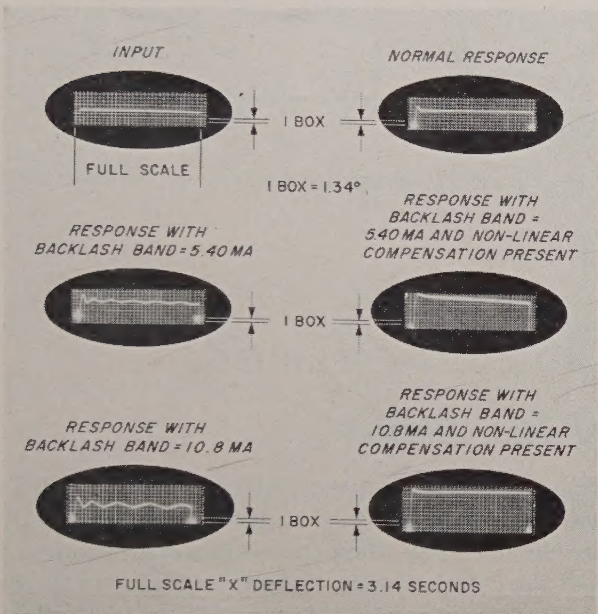

Fig. 13—High-speed analog computer results—servo no. 1.

If a nonlinear compensation device is incorporated into the second servo as shown in Fig. 14, then the basic condition for a limit cycle is

$$1 + NG_1G_2[G_3 + G_4G_5] + N'G_1G_5G' = 0. \qquad (10)$$

Variations in N' and G' are treated in a manner similar to that of the first servo. The shaded area of Fig. 5 represents a variation in the linear element G' where $G' = K(s+\alpha)/(s+\beta)$ as K varies from 0.225 to 0.898, α varies from 1.72 to 6.88, and β varies from 0.05 to 0.10. Nonlinear compensation of 15 per cent or more is required to eliminate the limit cycle. The network of Fig. 11 and a linear element $G' = 0.45(s+3.44)/(s+0.075)$ were incorporated into the system with the results shown in Fig. 15. An analog computer study was performed in order to show the effect of this nonlinear compensation network. The results are shown in Fig. 16.

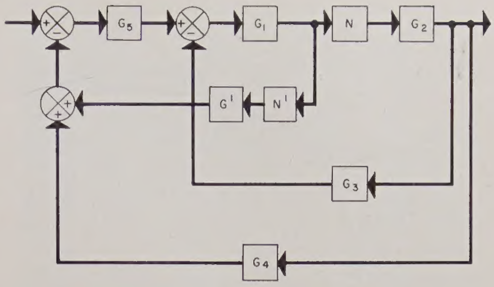

Fig. 14—Servo no. 2 with nonlinear compensation.

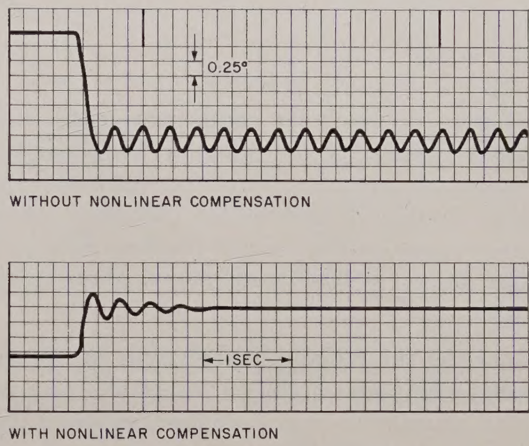

Fig. 15—Step response of servo no. 2.

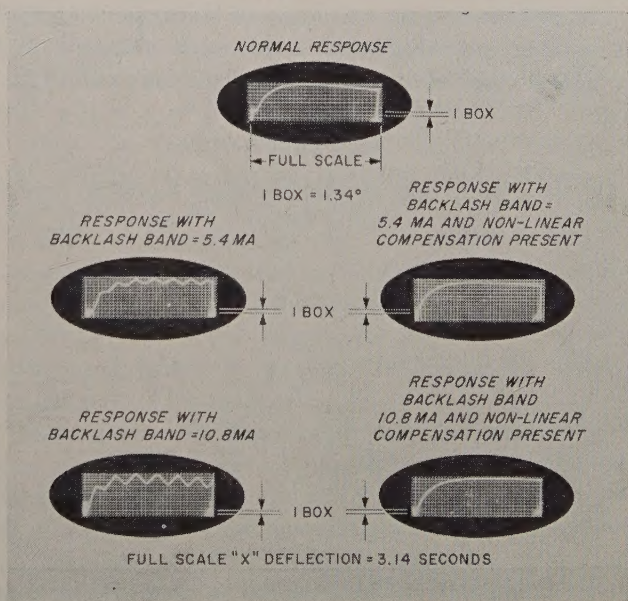

Fig 16—High-speed analog computer results—servo no. 2.

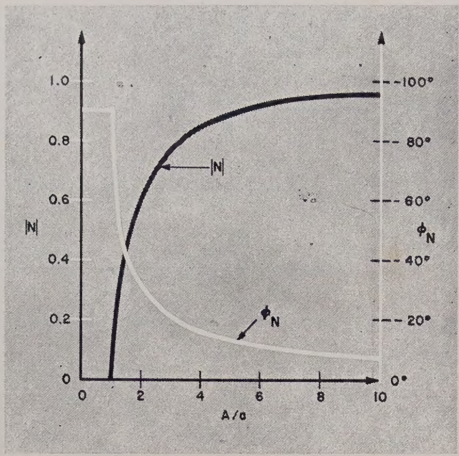

Fig. 17—Describing function for backlash.

Conclusion

The design of nonlinear compensation to eliminate oscillations caused by an inherent nonlinearity in a servo can be performed without too much difficulty by the use of the describing function technique. However, inherent errors exist in the use of the describing function technique due to the basic assumptions of the technique, which are:

1) Only the fundamental component of the Fourier series expansion of the output of the nonlinearity for a sinusoidal input is important since attenuation is present in the loop for harmonics of the fundamental frequency.
2) Only one nonlinearity exists in the system.

Nonlinear compensations of only 5 per cent and 15 per cent are required in the two servos analyzed in this paper. Because of the inaccuracy of the describing function technique and the variation of component nominal values over environmental conditions, a figure of 20 per cent is more desirable in order to insure stability.

Appendix I

The describing function for backlash has been derived in several papers, with the following results (Fig. 17):[6]

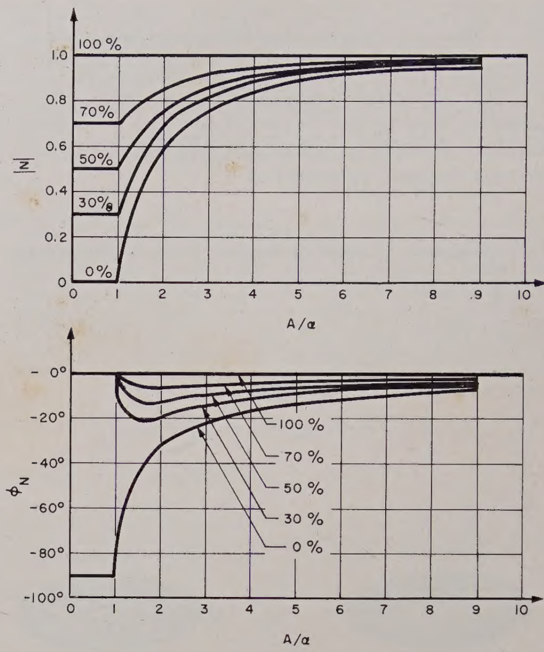

Fig. 18—Describing function for nonlinear-compensated backlash.

where

$u = 1-2a/A$
$e_{in} = A \sin t$
$2a =$ width of backlash.

When a nonlinear compensating element is inserted into a loop so that its input is the same as the existing nonlinear element, and when the frequency-dependent portion of the compensating element is similar to the existing frequency-dependent portion of the loop, then the effective nonlinear feedback element is $N+N'$, where N is the describing function for the existing nonlinear element and N' is the describing function for the compensating element. For $N'=x(1-N)$, the effective nonlinear feedback becomes $x+(1-x)N$ where x varies from 0 to 1.0 or 0 per cent to 100 per cent. Fig. 18 is a plot of $N+N'$ as x varies from zero per cent (no compensation) to 100 per cent (perfect compensation).

$$|N| = \frac{1}{\pi}\sqrt{1-u^2+\left(\frac{3\pi}{2}-\sin^{-1}u\right)^2+2\left(\frac{3\pi}{2}-\sin^{-1}u\right)u\sqrt{1-u^2}} \tag{11}$$

$$\phi_N = \tan^{-1}\frac{u^2-1}{\dfrac{3\pi}{2}-\sin^{-1}u+u\sqrt{1-u^2}} \tag{12}$$

[6] H. D. Greif, "Describing function method of servomechanism analysis applied to most commonly encountered nonlinearities," *Trans. AIEE*, pt. II, vol. 72, pp. 243–248; September, 1953.

APPENDIX II

The operation of the circuit of Fig. 11 to a sinusoidal input is the following: Initially the capacitor voltage remains zero until one of the diodes begins to conduct. At this time the capacitor voltage increases until the difference between the input and capacitor voltages becomes less than the voltage required to maintain diode conduction. The capacitor voltage will then remain constant until the difference between the input and capacitor voltages becomes greater than the voltage required to cause diode conduction. At this time, the operation cycle begins again with the other diode (see Fig. 19). The capacitor voltage response to a sinusoidal input is identical to the output response of a device having backlash.

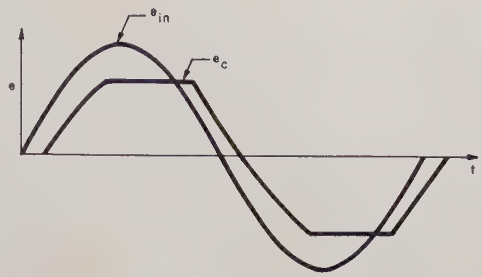

Fig. 19—Response of capacitor voltage of Fig. 11 for a sinusoidal input.

Specification of the Linear Feedback System Sensitivity Function*

WILLIAM M. MAZER†

Summary—**This paper examines the problem of designing a linear feedback system so that its response to a specified input is relatively insensitive to slow changes in system parameters. Classical feedback design techniques involve the specification of the system sensitivity function on the basis only of the forced response to a given input. A new performance criterion has been derived, in which the mean square variation of the system response is minimized. This specification of the sensitivity function results in control over the variation of both the characteristic and forced responses with system changes. The approach is based upon the assumption of a differential change in the variable system parameter, but yields workable results for large changes. It employs mathematical techniques which have been well developed for other applications in network design. The stability problem associated with high loop gains is considerably reduced when the sensitivity function is specified on the basis of this criterion.**

* Manuscript received by the PGAC, September 28, 1959; revised manuscript received, January 15, 1960. This paper is based on a dissertation submitted in partial fulfillment of requirements for the D.E.E. degree at the Polytechnic Institute of Brooklyn, Brooklyn, N. Y. The work described was done at the Microwave Research Institute of the Polytechnic Institute of Brooklyn, under Contract DA-30-069-ORD-1560 of the Office of Ordnance Research, and at the New York Systems Laboratory, Surface Communications Division, Defense Electronic Products.

† Defense Electronic Products, Radio Corp. of America, New York, N. Y.

INTRODUCTION

THIS SECTION is a summary of previous work in the area of feedback sensitivity.

The network designer synthesizes system structures which must respond in a specified manner despite changes in component behavior. Practically, the achievement of a desired response characteristic and its relative insensitivity to system parameter changes are both basic requirements of a successful design. In many cases, the invariability of the response is even more important than its nominal behavior. Analysis and synthesis procedures must take account of these requirements, with the precision afforded by available mathematical techniques.

Although purely passive, linear circuit elements very often satisfy a specified dynamic response requirement; the use of active elements in conjunction with feedback offers the possibility of extreme stability of this response characteristic. Furthermore, the necessity for performing energy conversions often makes the use of active elements mandatory. The diversity of reasons for using

feedback[1,2] may be summed up into two general areas, *i.e.*, convenience, and insensitivity of the system response. For example, reduction of the source impedance of a network by means of feedback may be regarded as a minimization of the sensitivity of the response characteristic to changes in the load impedance, which is viewed as part of the feedback system structure. Feedback may also be used to force an active element, or controller, to have a response considerably different from its open loop response. Convenience and response insensitivity, or "self-calibration," are simultaneously achieved by means of feedback when the characteristics of the control devices are known only as a statistical distribution, or within a specified tolerance envelope.

A quantitative definition of feedback sensitivity is required in order to enable precise system synthesis and analysis procedures to be established. The ultimate interest in time domain behavior makes the definition of a time domain sensitivity function appear to be desirable. However, analytic difficulties, as well as problems in formulating suitable specifications for such a function, make the use of the inverse of the complex frequency domain function, defined by Bode,[3] a more feasible approach. Although a considerable literature exists,[3,4] relating this sensitivity function to the elements of a feedback structure under steady-state sinusoidal excitation, all the advantages of the complex function theory are available to the designer, and system inputs of a more general nature may be considered.

Consider the linear, lumped parameter feedback structure shown in Fig. 1. The premultiplying transfer

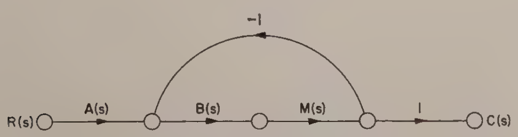

Fig. 1—Basic feedback system.

function $A(s)$ and the function $B(s)$ in the forward path are assumed to be compensating networks whose parameters are invariant. The transfer function $M(s)$ may be written

$$M(s) = k \frac{(s + Z_1)(s + Z_2) \cdots}{(s + P_1)(s + P_2) \cdots}. \tag{1}$$

[1] J. G. Truxal, "Automatic Feedback Control System Synthesis," McGraw-Hill Book Co., Inc., New York, N. Y., p. 113; 1955.

[2] E. J. Angelo, "Design of Feedback Systems," Microwave Res. Inst., Polytechnic Inst. of Brooklyn, Brooklyn, N. Y., Res. Rept. No. R-449-55, PIB-379.

[3] H. W. Bode, "Network Analysis and Feedback Amplifier Design," D. Van Nostrand Co., Inc., New York, N. Y.; 1945.

[4] S. Barabaschi and E. Gatti, "Modern methods of analysis of electrical linear active networks with particular regard to feedback systems," *Energia Nucleare*, vol. 2, pp. 104–119, December, 1954; vol. 2, pp. 168–197; February, 1955.

$M(s)$ is an active element, such as a controller or amplifier. The quantities Z_j and P_i are zeros and poles, respectively, of $M(s)$, and k is its gain. These elements are considered variables, with respect to which the system response is to be stabilized; they are assumed to change slowly compared to the system response time. The closed loop transmission function is given by

$$T(s) = \frac{C(s)}{R(s)} = \frac{A(s)B(s)M(s)}{1 + B(s)M(s)}. \tag{2}$$

$C(s)$ and $R(s)$ are the transforms of the system output and input, respectively. With respect to a variable element x of $M(s)$, the sensitivity of $T(s)$ is defined as[5]

$$S_x^T(s) = \frac{x}{T(s)} \frac{dT(s)}{dx} = \frac{d \ln T(s)}{d \ln x} = \frac{dT/T}{dx/x} \tag{3}$$

Thus the changes in $T(s)$ and x have been normalized with respect to their nominal values. By suitable transformations, the premultiplying transfer function may be eliminated, and a feedback structure with a compensation network in the feedback path achieved. Alternatively, the forward path compensating network inside the loop may be eliminated. The results obtained here will be applicable to such structures as well. Differentiating (2), and inserting the result into (3), one obtains

$$S_x^T(s) = \frac{x}{M(s)} \frac{dM(s)/dx}{1 + B(s)M(s)}$$

$$= \frac{d \ln M(s)}{d \ln x} \frac{1}{1 + B(s)M(s)}. \tag{4}$$

For $x = k$, Z_j, or P_i, (4) becomes

$$S_k^T(s) = \frac{1}{1 + B(s)M(s)} \tag{5}$$

$$S_{Z_j}^T(s) = \frac{Z_j}{s + Z_j} \frac{1}{1 + B(s)M(s)} \tag{6}$$

$$S_{P_i}^T(s) = \frac{-P_i}{s + P_i} \frac{1}{1 + B(s)M(s)}. \tag{7}$$

The results given by (5)–(7) may be obtained in an alternate manner. In accordance with Bode's theorem, whereby any transmission $T(s)$ may be written as a bilinear transformation of the parameter x, the diagram of Fig. 1 is redrawn into the forms given in Fig. 2. The variable element appears only in the forward path within the feedback loop in each case. Fig. 2(d) illustrates

[5] The inverse of Bode's sensitivity function.

Suppose $M = K \frac{s + z_j}{s + P_i}$

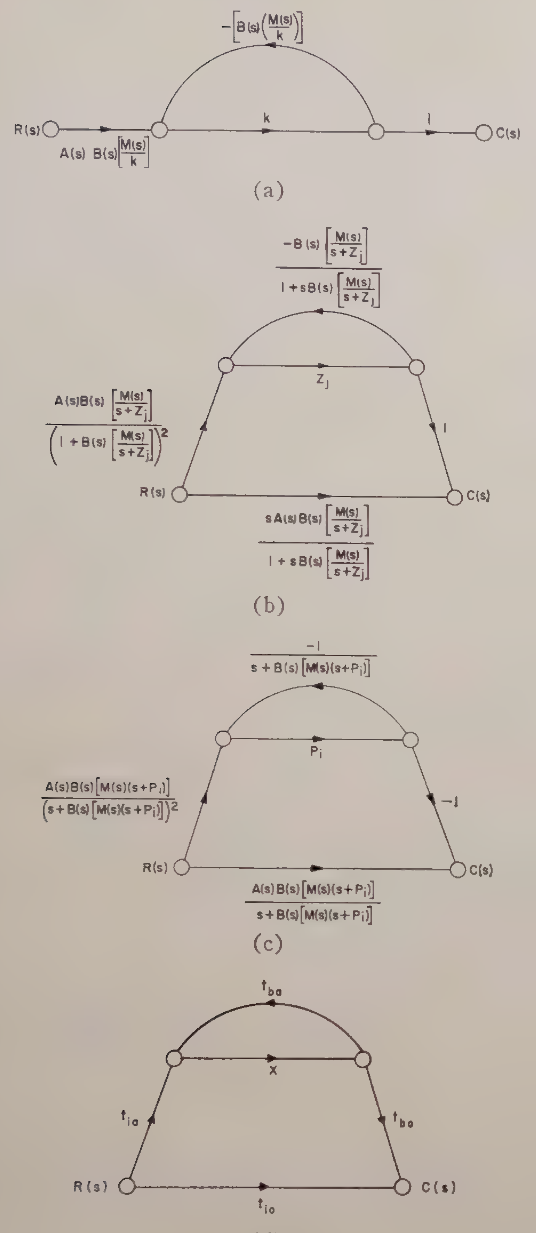

Fig. 2—Alternate system configurations.

the notation used by Truxal,[6] for the parameter x. Using this notation,

$$S_x^T(s) = \frac{x t_{ia} t_{b0}}{T(1 - x t_{oa})^2} = \frac{1}{F_x}\left[1 - \frac{t_{i0}}{T}\right]. \tag{8}$$

Here F_x is the return difference with respect to the parameter x. The functional dependence on s has been omitted. The results given by (5)–(7) may be obtained directly using (8).

[6] Truxal, *op. cit.*, p. 120 ff.

The return difference for the zero transmission leakage case, where k is the variable parameter, is a common factor among these sensitivity functions. This results in a definite relationship among the system response variations, due to each variable parameter.

The sensitivity with respect to k approaches unity as s approaches infinity. This may be ascertained from (5), where it is evident that $B(s)M(s)$ approaches zero at high frequencies, if this product corresponds to a physical structure. The system operates open loop at high frequencies, so that the relative variation of $T(s)$ is identical with that of k. Thus $S_k^T(s)$ has an equal number of finite poles and zeros. The sensitivities with respect to Z_j and P_i have an added finite pole.

Although the sensitivity function has been defined and several of its properties examined for the particular system model under consideration, it is not yet apparent how one may utilize it in system design, or more fundamentally, on what basis its characteristics should be chosen. The following sections consider various bases for the specification of the sensitivity function.

Specification of the Sensitivity Function

A common design approach consists of making the "loop gain," or return difference, extremely large in the band of frequencies in which the input signals are expected to lie. Then the open loop response is caused, by appropriate shaping, to fall off as rapidly as possible to unity gain outside this band. A maximum roll-off rate of about 33 db per decade permits unconditional stability to be achieved, while a greater rate results in conditional stability. Thus realization of low reponse sensitivity by making the magnitude of the sensitivity function small in the desired band of frequencies inevitably results in the problem of system stability. Yet this technique often yields a considerably overdesigned system, particularly when the input signal is known to be a restricted type, such as a succession of steps or decaying exponentials. Although it is usually desirable to restrict the variability of the transient, as well as the steady-state response, with changes in system parameters, this approach is analytically sound only for variations of the forced response to sinusoidal excitations.

On the other hand, a synthesis procedure is available in which the problems of system stability and sensitivity are theoretically divorced.[7–9] Essentially, it can be shown that the inclusion of two compensation networks in a conditionally stable feedback system enables the separate specification of the system transmission and

[7] C. Lang and J. M. Ham, "Conditional feedback systems—a new approach to feedback control," *Trans. AIEE*, pt. II, vol. 74, pp. 152–161; July, 1955.

[8] J. G. Truxal, "Modern network theory and its application to feedback control," *Proc. Conf. on Systems Engrg.*, Purdue University, Lafayette, Ind., pp. 79–104; July, 1955.

[9] R. J. D. Reeves, "Feedback amplifier design," *Proc. IEE*, pt. IV, vol. 99, pp. 383–389; October, 1952.

sensitivity functions. To design a system, such as shown in Fig. 1, to meet only a prescribed transmission function requirement, any number of combinations of $A(s)$ and $B(s)$ may be employed. However, if a specified sensitivity function is to be achieved as well, then $A(s)$ and $B(s)$ are uniquely defined. Additional problems are usually encountered in the physical realization of the desired compensation networks. However, suitable approximation procedures may be employed to yield workable results, although the desired transmission and sensitivity functions will not be realized exactly.

The possibility of achieving low system sensitivity with small values of loop gain has been discussed in the literature,[10,11] although the inputs dealt with were steady-state sinusoids. The basic problem is the determination of criteria for the specification of the sensitivity function for an arbitrary input. After the form of the desired transmission function has been ascertained, based on some dynamic response criterion, (5)–(7) reveal that the system parameters which remain to be specified are the zeros of the sensitivity function. The poles of the sensitivity and transmission functions are identical, except for the added poles of Z_j and P_i, provided the product $A(s)B(s)M(s)$ in (2) is chosen so that no poles are added to those of the reciprocal of $F_k(s)$.

Based upon this approach to the feedback system synthesis problem, an entirely new procedure for specifying the sensitivity function may be formulated for an arbitrary input. In the following section, results previously achieved in this area are discussed. In the succeeding sections, the major contributions of this paper are presented.

The Forced Response Criterion

Suppose the desired response in the time domain is given by $c(t)$, and the response error by $e(t)$. Then the zeros of $S(s)$ must be determined so that $e(t)$ behaves in a desired manner over the interval or intervals of interest in the time domain. A considerable literature exists whereby one may relate time domain responses to the complex frequency domain characteristics of a given network configuration.[12–14] These results are useful when the inputs are restricted types, and when the system poles are simple. Functions of third or higher order, with multiple poles, and possible poles on the $j\omega$ axis, are difficult to deal with. Since the steady-state error may be determined without actually calculating $c(t)$, this approach would be of interest in controlling $e(t)$ over the part of the time response during which the transient is still appreciable. The computational difficulties inherent in establishing a quantitative relationship between time response and the complex frequency characteristics of the sensitivity function preclude such a procedure as a general design technique.

If the sensitivity function is to be specified only on the basis of the forced response to a given input, then the zeros of $S(s)$ are chosen as follows. If $R(s)$ consists of a number of inputs of the form

$$R_k(s) = a_k \frac{1}{s + s_k}, \tag{9}$$

then the system output is given by

$$C_k(s) = a_k \frac{1}{s + s_k} T(s), \tag{10}$$

and the forced response is

$$c_k(t) = a_k T(-s_k) e^{-s_k t}. \tag{11}$$

Taking the derivative of $c_k(t)$ with respect to a variable parameter x, and substituting (3):

$$\frac{dc_k(t)}{dx} = \frac{a_k}{x} T(-s_k) S_x^T(-s_k) e^{-s_k t}. \tag{12}$$

Thus, if the sensitivity function has a zero at the pole of the transform of the input function, the system forced response is invariant with changes in x. A number of zeros equal to the system order is available for selection on this basis. If the s_k are expected to lie in some known range of values, the sensitivity zeros may be distributed accordingly. This result is valid when s_k is imaginary, but in that case both the steady-state amplitude and phase responses are of interest. For a sinusoidal input, the forced response and its derivative with respect to x become:

$$c_r(t) = \frac{a_r}{s_r} \left| T(js_r) \right| \sin \left[s_r + \phi(is_r) \right] \tag{13}$$

where

$$\phi(js_r) = \tan^{-1} \frac{\operatorname{Im} T(js_r)}{\operatorname{Re} T(js_r)}$$

$$\frac{dc_r(t)}{dx} = \frac{a_r}{s_r} \left\{ \sin (s_r t + \phi) \frac{d\left| T(js_r) \right|}{dx} + \left| T(js_r) \right| \frac{d\phi(js_r)}{dx} \cos (s_r t + \phi) \right\}. \tag{14}$$

[10] H. S. Black, "Stabilized feedback amplifiers," *Bell. Sys. Tech. J.*, vol. 13, pp. 1–18; January, 1934.

[11] D. L. H. Gibbings and A. M. Thompson, "Gain stability of feedback amplifiers," *Proc. IEE*, pt. III, vol. 101, pp. 35–37; January, 1954.

[12] J. H. Mulligan, Jr., "The effect of pole and zero locations on the transient response of linear dynamic systems," PROC. IRE, vol. 37, pp. 516–529; May, 1949.

[13] W. H. Kautz, "Transient synthesis in the time domain," IRE TRANS. ON CIRCUIT THEORY, vol. CT-1, pp. 29–39; September, 1954.

[14] I. Gumowski, "Some relations between frequency and time domain errors in network synthesis problems," IRE TRANS. ON CIRCUIT THEORY, vol. CT-5, pp. 66–69; March, 1958.

Writing

$$T(js_r) = |T(js_r)|\, e^{j\phi}, \tag{15}$$

then

$$\begin{aligned} S_x^T(js_r) &= \frac{x}{|T|\, e^{j\phi}} \frac{d[|T|\, e^{j\phi}]}{dx} \\ &= \frac{x}{|T|} \frac{d|T|}{dx} + jx\frac{d\phi}{dx} \\ &= S_R(js_r) + jS_I(js_r). \end{aligned} \tag{16}$$

The R and I subscripts in the last equation refer to the real and imaginary parts of $S(js_r)$, respectively. Thus, if zeros of $S(s)$ are assigned at $\pm js_r$, then both the forced amplitude and phase responses are invariant with changes in x, as seen in (14). However, if these zeros are assigned only to the even or odd parts of $S(s)$, then only the amplitude or phase response, respectively, will be invariant wlth changes in x, although $S(s)$ itself will have no zeros at $\pm js_r$. In general the zeros of $S(s)$ will then be complex, which should simplify the approximation problem.

This section has demonstrated how variations in the forced response may be controlled by suitably specifying the sensitivity function. The lack of control over variations in the transient response is conspicuous. This paper adds to prior work by developing a relationship between the variability of the complete response, including the transient, and the sensitivity function. The remaining part of the paper develops these ideas in the form of a suggested design procedure.

The Minimum Mean Square Error Criterion

A design based only upon the forced response is likely to result in an unsuitable selection of the zeros of the sensitivity function, if the system is of high order, and the input is of a known, invariant form. For example, if a fifth-order system is to be subjected to a sinusoidal input, then five sensitivity zeros must be assigned. Two zeros are given values corresponding to the poles of the input transform; there is no reasonable basis for selecting the remaining three. Furthermore, if interest is centered upon the part of the system response where the transient has an appreciable value, then even the forced response criteria outlined have little value in determining suitable sensitivity zeros.

The design approach based upon root locus techniques results in the choice of a network configuration with poles and zeros which move in such a way, with a given system parameter change, that the basic character of the transient response remains unchanged. A different point of view may be taken in employing a curve matching technique, with the specification of the sensitivity zeros as the end product of the procedure. One common curve matching technique minimizes the mean square difference between a desired and an actual response. A straightforward mathematical technique has already been developed for the synthesis of feedback systems, and for the design of filters with noisy inputs, based upon a minimum mean square error criterion. This technique is directly applicable to the sensitivity problem; its chief advantage is that the problem is solved entirely in the complex frequency domain. Thus the necessity for obtaining the error in the time responses of high-order feedback systems, in terms of the unknown zeros of the sensitivity function, is avoided. Before proceeding with a detailed exposition of this method, the following definitions are required:

$$\begin{aligned} T(s) &= \frac{N(s)}{D(s)}, \\ S(s) &= \frac{N_0(s)}{D(s)}, \\ \Delta T(s) &= \frac{\Delta x}{x} T(s)S(s), \\ E(s) &= R(s)\Delta T(s). \end{aligned} \tag{17}$$

The Δ notation is used to indicate a small finite, rather than differential, change. The mean square error is given by

$$\overline{e^2} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} e^2(t)dt. \tag{18}$$

Here $e(t)$ is the inverse transform of $E(s)$. The square of the error is integrated over a time interval of length $2T$, its average is taken, and the time interval is then allowed to become infinite. By a simple mathematical exercise it may be shown that[15]

$$\overline{e^2} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Delta T(s)\Delta T(-s)\Phi_r(s)ds. \tag{19}$$

$\Phi_r(s)$ is the power density spectrum of the input signal. Substituting (17) into (19),

$$\overline{e^2} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \left(\frac{\Delta x}{x}\right)^2 \cdot \frac{N_0(s)N_0(-s)}{D(s)D(-s)} T(s)T(-s)\Phi_r(s)ds. \tag{20}$$

To minimize the mean square error $\overline{e^2}$ one takes its partial derivatives with respect to each unknown coefficient of $N_0(s)$, and sets the results equal to zero. This yields a number of linear algebraic equations equal to the number of unknown coefficients. Furthermore, if $N_0(s)$ is of even (odd) degree, all coefficients of odd (even) powers of s must equal zero.

[15] Truxal, *op. cit.*, footnote 1, p. 411 ff.

In order to obtain a suitable power density spectrum for use in (20), aperiodic inputs such as step and ramp functions, with which the feedback system designer is accustomed to deal, must be modified by being repeated periodically. Although the system is then designed on the basis of this fictitious input, it may be shown that the results obtained are of considerable value when the input is aperiodic. Furthermore, the periodicity of the repeated input signal determines approximately the interval of time over which the mean square error, for the case of an aperiodic input, is minimized. The system may also be optimized with respect to stationary random inputs.

Another possible approach is the use of an integrated square instead of a mean square error minimization criterion. This appears to offer the advantage that the true aperiodic input is used in the synthesis. This procedure would involve expressing the time domain response variation in terms of the unknown zeros of the sensitivity function. This results in an unwieldy and complicated design procedure. The advantages of working entirely in the complex frequency domain are lost. At least one exception occurs with the use of a weighting function, such as a decaying exponential function, which results in a straightforward synthesis procedure involving only the evaluation of residues.

The step-by-step solution of a typical problem will clarify the minimum $\overline{e^2}$ design procedure described above.

Design Example

Design a feedback system in the configuration of Fig. 1, with the following closed loop transfer function:

$$T(s) = \frac{5}{(s^2+s+1)(s+5)}. \tag{21}$$

Since the system is of the third order, $N_0(s)$ may immediately be written

$$N_0(s) = s^3 + a_2 s^2 + a_1 s + a_0. \tag{22}$$

If the variable element is the forward gain k, the mean square error becomes

$$\overline{e^2} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(\frac{\Delta k}{k}\right)^2 \frac{T(s)T(-s)}{D(s)D(-s)} \cdot [(a_2 s^2 + a_0)^2 - s^2(s^2+a_1)^2]\Phi_r(s)ds. \tag{23}$$

Since $\overline{e^2}$ is the sum of two integrals, one may solve for a_2 and a_0 separately. Taking partial derivatives with respect to these quantities,

$$\frac{\partial \overline{e^2}}{\partial a_2} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(\frac{\Delta k}{k}\right)^2 \frac{T(s)T(-s)}{D(s)D(-s)} \cdot 2(a_2 s^2 + a_0)s^2\Phi_r(s)ds = 0 \tag{24}$$

$$\frac{\partial \overline{e^2}}{\partial a_0} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(\frac{\Delta k}{k}\right)^2 \frac{T(s)T(-s)}{D(s)D(-s)} \cdot 2(a_2 s^2 + a_0)\Phi_r(s)ds = 0. \tag{25}$$

Solving (24) and (25) for a_2 and a_0, an homogeneous set of equations results, the solutions of which are $a_2 = a_0 = 0$. To solve for a_1,

$$\frac{\partial \overline{e^2}}{\partial a_1} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(\frac{\Delta k}{k}\right)^2 \frac{T(s)T(-s)}{D(s)D(-s)} \cdot [-2s^2(s^2+a_1)]\Phi_r(s)ds = 0 \tag{26}$$

$$a_1 = \frac{\dfrac{1}{2\pi j}\displaystyle\int_{-j\infty}^{j\infty}\frac{T(s)T(-s)}{D(s)D(-s)}(s)^2(-s^2)\Phi_r(s)ds}{\dfrac{1}{2\pi j}\displaystyle\int_{-j\infty}^{j\infty}\frac{T(s)T(-s)}{D(s)D(-s)}(s)(-s)\Phi_r(s)ds}. \tag{27}$$

Thus a_1 is independent of Δk, provided the latter is small enough for the first-order approximation $\Delta T(s)$ given by (17) to be reasonably valid. Furthermore, $\Phi_r(s)$ being positive, a_1 is always positive, and the integrands are written as conjugate products in (27) to emphasize this fact.

The character of the input remains to be specified, so that $\Phi_r(s)$ may be evaluated. Suppose the input is a sinusoid of frequency $\beta = \frac{1}{2}$ rad/second, and that the mean square error in the system response, over one cycle of the driving function, is to be minimized for small changes in k. To generate an appropriate repetitive input signal, let the input sinusoid be multiplied by a square wave of frequency $\omega_0 = \frac{1}{4}$ rad/second, and unity height. The wave depicted in Fig. 3 results. The desired output over $T_0/2 = 12.56$ seconds is shown in Fig. 4. The Fourier series and power spectrum of the square wave are

$$r_{sq}(t) = \frac{4}{\pi}\left(\sin\frac{1}{4}t + \frac{1}{3}\sin\frac{3}{4}t + \frac{1}{5}\sin\frac{5}{4}t + \cdots\right) \tag{28}$$

$$\Phi_{sq}(\omega) = \frac{1}{4}\left(\frac{4}{\pi}\right)^2\left\{\left[\delta\left(\omega-\frac{1}{4}\right)+\delta\left(\omega+\frac{1}{4}\right)\right] + \frac{1}{9}\left[\delta\left(\omega-\frac{3}{4}\right)+\delta\left(\omega+\frac{3}{4}\right)\right] + \frac{1}{25}\left[\delta\left(\omega-\frac{5}{4}\right)+\delta\left(\omega+\frac{5}{4}\right)\right] + \cdots. \tag{29}$$

Here $\delta(\omega \pm n\omega_0)$ is the Dirac delta function. If this square wave is driven through a fictitious transfer function,

$$F(s) = \frac{s}{s^2+\beta^2} = \frac{s}{s^2+(\frac{1}{2})^2} \tag{30}$$

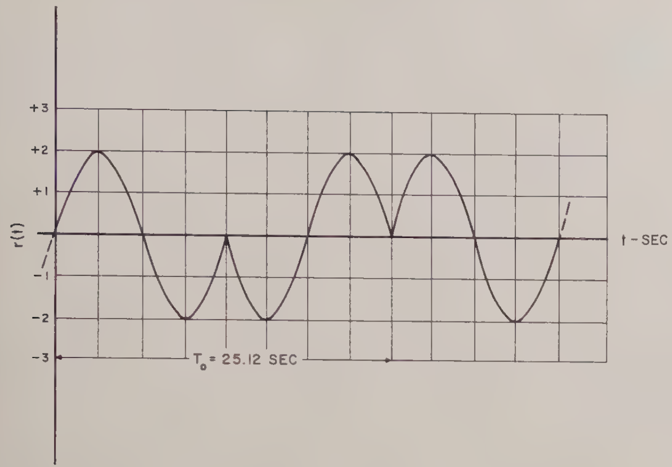

Fig. 3—Input function.

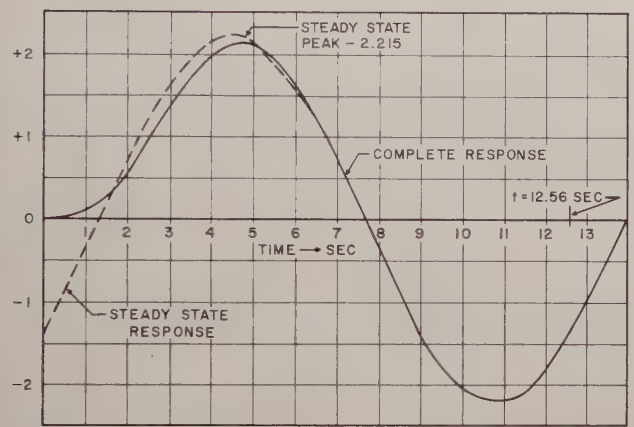

Fig. 4—Plot of $c_0(t)$ vs time. Response of

$$T_0(s) = \frac{5}{(s+5)(s^2+s+1)}$$

to a sinusoid with transform

$$\frac{1}{s^2+(1/2)^2}.$$

then the wave shown in Fig. 3 results. The power spectrum of the input is thus given by

$$\Phi_r(\omega) = \Phi_{sq}(\omega) \left| F(j\omega) \right|^2$$

$$= \frac{\omega^2}{[\omega^2 - (\frac{1}{2})^2]^2} \Phi_{sq}(\omega) \tag{31}$$

with $\Phi_{sq}(\omega)$ given in (29). The integrals of (27) are evaluated by summing twice the values of the integrands at $\omega = \frac{1}{4}, \frac{3}{4}, \frac{5}{4}$, etc., since they are even functions and thus always positive. The summations are continued until the desired accuracy is achieved. The values of the integrands fall rapidly with increasing ω, since the degrees of the denominators are much larger than those of the numerators. Substituting $T(s)$, $D(s)$, and $\Phi_r(s)$, one obtains

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{T(s)T(-s)}{D(s)D(-s)}(s)^3(-s)^3\Phi_r(s)ds = 0.751(10^{-2}) \tag{32a}$$

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{T(s)T(-s)}{D(s)D(-s)}(s)^2(-s)^2\Phi_r(s)ds = 1.238(10^{-2}) \tag{32b}$$

$$\frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{T(s)T(-s)}{D(s)D(-s)}(s)(-s)\Phi_r(s)ds = 2.511(10^{-2}). \tag{32c}$$

Then by (27),

$$a_1 = 0.494. \tag{33}$$

This value of a_1 is accurate to 0.2 per cent using only the first five terms in the Fourier series of (29). Now the mean square error is

$$\overline{e^2} = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\left(\frac{\Delta k}{k}\right)^2 \frac{T(s)T(-s)}{D(s)D(-s)} \cdot (-s^2)(s^2+a_1)^2\Phi_r(s)ds. \tag{34}$$

Substituting (32) and (33) into (34),

$$\overline{e^2} = 0.00149\left(\frac{\Delta k}{k}\right)^2. \tag{35}$$

The sensitivity function is

$$S_k^T(s) = \frac{s(s^2+0.494)}{(s^2+s+1)(s+5)}\cdot \tag{36}$$

A comparison with the forced response criteria, in terms of relative values of $\overline{e^2}$ and other factors, is useful to assess the merit of this approach.

Comparison of Designs

To insure zero variation of both the steady-state amplitude and phase responses, one would have set $a_1 = \beta^2 = \frac{1}{4}$. For this value of a_1, $\overline{e^2}$ becomes

$$\overline{e^2} = 0.00269\left(\frac{\Delta k}{k}\right)^2. \tag{37}$$

The minimum $\overline{e^2}$ design thus reduces this mean square error by a factor of 1.8. A design in which only the even part of $S(s)$ is assigned the zeros of $s = \pm j\frac{1}{2}$ yields only steady-state amplitude response invariability. In this case, let

$$S_R(s) = \frac{-s^2[s^2+(1/2)^2]^2}{(s^2+s+1)(s+5)(s^2-s+1)(-s+5)}\cdot \tag{38}$$

Now since

$$S_R(s) = 1/2[S(s) + S(-s)], \tag{39}$$

therefore

$$S(s) = \frac{s(s^2 + 1.067s + 0.899)}{(s^2 + s + 1)(s + 5)}. \tag{40}$$

Substituting (40) into (20),

$$\overline{e^2} = 0.0197\left(\frac{\Delta k}{k}\right). \tag{41}$$

Comparing (41) with (35), the mean square error is reduced by a factor of 13.

The results achieved in this example of a third-order system indicate that the zeros of $N_0(s)$ occur at $s=0$, and on the $j\omega$ axis in conjugate pairs. This may be shown to be true regardless of system order. To examine the influence on $\overline{e^2}$ of a nonzero coefficient of s, say δ, let $N_0(s)$ be given by

$$N_0(s) = s(s^2 + \delta s + 0.494). \tag{42}$$

Then

$$N_0(s)N_0(-s) = -s^2[(s^2 + 0.494)^2 - \delta^2 s^2]. \tag{43}$$

The first factor within the bracket results in the minimum $\overline{e^2}$ given by (35). The $\overline{e^2}$ corresponding to the second factor has been evaluated in (32b). Thus the new $\overline{e^2}$ is

$$\overline{e^2} = [0.00149 + 0.0124\delta^2]\left(\frac{\Delta k}{k}\right)^2. \tag{44}$$

The value of δ which causes $\overline{e^2}$, given by (44), to be double that of (35) is given by

$$\delta = \left[\frac{0.00149}{0.0124}\right]^{1/2} \approx 0.35. \tag{45}$$

This results in a very severe requirement on the R/L ratio of the system compensation circuitry. However, the transmission function in the design example has an undamped natural frequency at $\omega_n = 1$ rad/second, and δ is normalized with respect to this quantity. When $\omega_n = 100$, $\delta = 35$, which is a considerably relaxed circuit R/L requirement. Thus the practical application of the minimum $\overline{e^2}$ design appears to be limited to systems of considerable bandwidth.

Although all calculations required to obtain $N_0(s)$ have been performed in the complex frequency domain, it is instructive to check the results obtained by additional calculations in the time domain. Ordinarily these are not a necessary part of the design procedure. For example, it would be of interest to determine the effect on the transient response of a finite change in k, since an infinitesimal change was assumed throughout the calculations. Furthermore, since the system was designed for the input given by Fig. 3, the transient response and the mean square error resulting from a suddenly applied sine wave with no phase reversals, should be examined. The time domain response errors, for systems designed on the basis of different criteria, should be compared.

The first step in performing exact error response calculations involves solving for $A(s)$ and $B(s)$ in Fig. 1, in terms of the specified nominal transmission and sensitivity functions, when k has some nominal value $k = k_0$. Thus the compensation networks will contain models of these nominal functions. Substituting these values of $A(s)$ and $B(s)$ into (4), the transmission for any value of k becomes

$$T(s) = \frac{1}{1 + \left[\frac{k_0}{k} - 1\right]S(s)} T_0(s). \tag{46}$$

Here $T_0(s)$ is the value of the transmission when $k = k_0$, *i.e.*, the desired transmission function. Substituting for $T_0(s)$ the value given in (21) and for $S(s)$, (36) and (40), and letting $k/k_0 = 0.8$, $T(s)$ becomes

$$T_e(s) = \frac{4}{(s + 3.785)(s^2 + 1.0155s + 1.057)} \tag{47a}$$

$$T_R(s) = \frac{4}{(s + 4.023)(s^2 + 0.9904s + 0.9954)}. \tag{47b}$$

The subscript e refers to the design based upon the minimum $\overline{e^2}$ criterion; the subscript R refers to the design in which the real part of $S(s)$ is specified. The responses of these transmission functions, and that given by (21), to suddenly applied sinusoids of amplitude $=2$, and angular frequency $\beta = \frac{1}{2}$, are plotted in Figs. 4 through 7. The squared error responses have been averaged over 12.56 seconds. From (35), the minimum $\overline{e^2}$ for the phase-reversed sinusoidal input is

$$\overline{e^2}_{\min} = 0.00149\left(\frac{\Delta k}{k}\right)^2 = 9.31(10^{-5}). \tag{48}$$

The value of $\overline{e^2}$ measured in Fig. 6, for the suddenly applied sinusoid, is

$$\overline{e^2}_{\text{meas}} = 7.82(10^{-5}). \tag{49}$$

Thus the measured value is lower than the calculated value of $\overline{e^2}_{\min}$ by about 15 per cent. Although part of this discrepancy may be accounted for by measurement and computational errors, there is no doubt that the periodicity of the input in the original design, as well as the finite change in forward gain, contribute to the error. Its value is small enough, however, considering the large factor by which $\overline{e^2}$ has been reduced, to provide a sound justification for this design technique. Furthermore, examination of Fig. 5 reveals that the

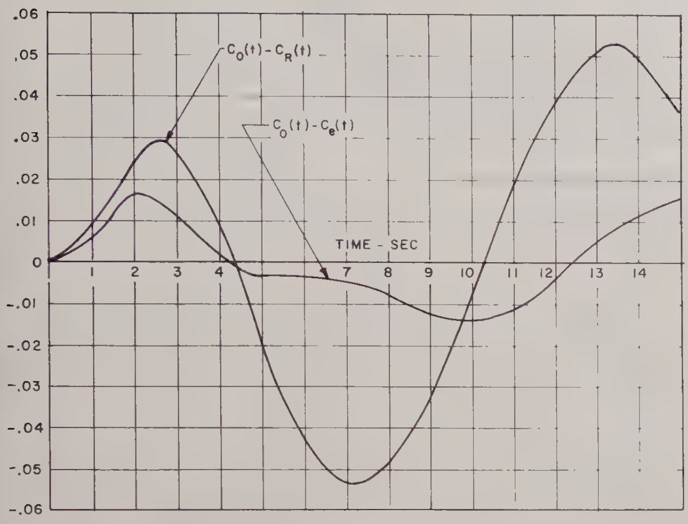

Fig. 5—Plots of $c_0(t)-c_e(t)$ and $c_0(t)-c_R(t)$ vs time.

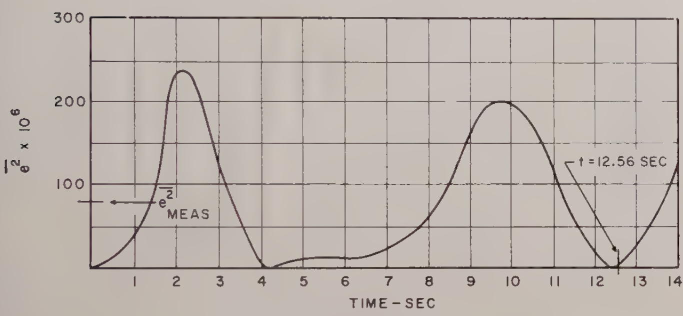

Fig. 6—Plot of $[c_0(t)-c_e(t)]^2$ vs time.

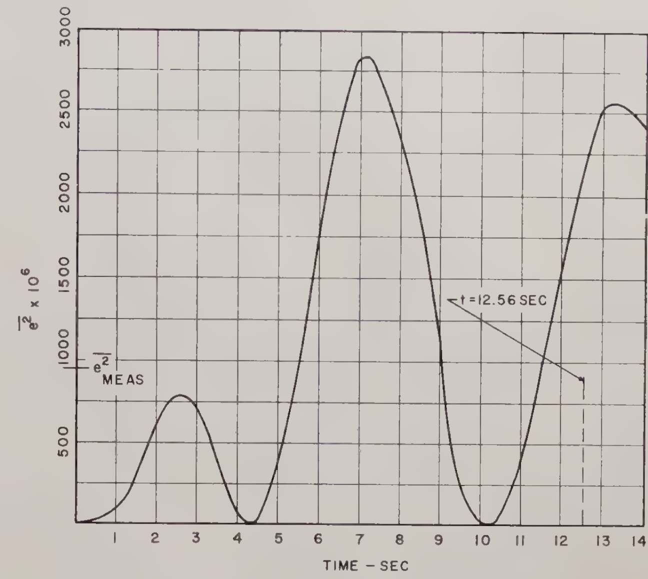

Fig. 7—Plot of $[c_0(t)-c_R(t)]^2$ vs time.

excursion of the system response from its desired value has been considerably reduced, compared to a design in which the transient response was ignored, without causing the character of the response to change in an unusual manner.

The design procedures for the determination of the sensitivities with respect to Z_j or P_i are similar to those outlined in this example. A pair of symmetrical poles on the real axis is added to the integrand in (20). If these do not significantly reduce the values of the integrand for the harmonics of ω_0 necessary to evaluate accurately the coefficients of $N_0(s)$, then the design for $x=k$ is optimum as well for $x=Z_j$ or $x=P_i$.

The zeros of the sensitivity function become poles of the open loop gain $B(s)M(s)$. With increasing loop gain, the root loci exhibit a small excursion into the right-half s plane. Thus the system is conditionally stable, but for the third-order system under consideration, has a gain margin of 21 db.

Conclusions

A mathematical definition of feedback sensitivity has been formulated. It has been shown that a feedback system sensitivity function may be so chosen that the system's forced response to a specified input is invariable with parameter changes. A mean square error criterion may be utilized to determine a suitable sensitivity function for minimizing changes in the system's transient response. Although the input function must be put into the form of a repetitive signal, in order to carry out the required procedure, the resulting design has been shown to have considerable merit for an aperiodic input, in a specific design example. The system so designed is conditionally stable, and has an open loop gain with poles very close to the imaginary s axis.

This new criterion is intended to be introductory to the more general study of relationships between variations of over-all system response and the sensitivity function. For example, the minimization procedure may be rederived, based upon various performance constraints. Multivariate designs may be developed. Minimum integrated square and other error criteria have extended the results of this paper.

Acknowledgment

The advice and encouragement of Dr. J. G. Truxal, Head of the Department of Electrical Engineering of the Polytechnic Institute of Brooklyn, and Dr. I. Horowitz of the Hughes Aircraft Corporation, are gratefully acknowledged.

Synthesis of Linear, Multivariable Feedback Control Systems*

ISAAC M. HOROWITZ†

Summary—A multivariable controlled process or plant is one in which there are n independent inputs and m outputs with $n>1$ and $m \leq n$. A control problem may exist for one or two principal reasons. 1) The plant parameters may vary or they may be only vaguely known, and the system response sensitivity to the parameter variation is to be reduced. 2) The system response to disturbances is to be reduced. A synthesis procedure for attaining these objectives and simultaneously realizing a desired set of system transmission functions is developed in this paper. The role of system configuration is considered. Design is broken up into two separate regions. In the significant system-response frequency region, there is straightforward synthesis in attaining the design objectives. In the higher frequency range, the loop transmission must be shaped so that the system is stable. The latter problem is considerably more difficult when there are substantial plant parameter variations. Some procedures are illustrated by two detailed examples ($n=m=2$ in one example, and $n=3$, $m=2$ in the second) in which there are large plant parameter variations.

Introduction

A MULTIVARIABLE (or multipole[1]) controlled process or plant is one in which there are n independent inputs, $x_1, x_2, \cdots, x_n$, and m outputs $c_1, c_1, \cdots, c_m$, with n greater than one, and m generally, but not necessarily, greater than one. Examples of multivariable plants are the control systems of turbojet engines, of airframes, and of steel rolling mills. The following problem is treated. Consider a given multivariable plant with its input-output relations expressed by an $m \times n$ matrix of transfer functions. It is assumed that the control engineer must work with the plant as it stands and cannot change its parameters and that the output cannot be influenced except through the plant. A control problem exists for one or two principal reasons.

1) The plant parameters are subject to variation. Typical reasons for such variation are changes in environmental conditions or in quiescent operating points; changes inherent in the nature of the plant, such as a change in mass due to fuel consumption, etc. It is assumed that if there are p significant plant parameters, it is possible to fix a closed region in p-dimensional space such that over the range of life and operation of the plant, the parameters may be found in that region.

Ignorance of the plant parameters may be included under this heading provided that, as in the above, a bounded region of plant behavior may be fixed. However, such ignorance cannot be too great. At least the physical capabilities must be known so that the plant is not expected to do what it is unable to do.

2) The second principal reason for the existence of a control problem is that the plant is subject to external disturbances and it is desired to control the effect of such disturbances. Here, too, at least a closed region of possible range and characteristics of the disturbances must be known.

This article is an extension of the work done by several authors[1–4] in which the principal results reported in the literature have been concerned with the synthesis of a desired over-all set of system transfer functions. In addition this earlier work is subject to some constraints such as noninteraction,[2,3] use of a diagonal loop-transmission-matrix,[1] and the assumption of small parameter variations[4] to simplify the stability problem. The purpose of this paper is to consider simultaneously how one can realize a matrix of system functions and an insensitivity of these to plant parameters variations, and/or a specified degree of system insensitivity to external disturbances. It may be argued that were it not for the latter two problems, there would be no need to apply feedback around the plant. A set of system transfer functions alone may be obtained much more cheaply and with considerably less design effort by cascading the plant with suitable prefilters. The treatment of the problem in this paper is subject to one serious limitation; *i.e.*, the relations between the inputs to the plant and the resulting outputs are linear. It is known that even for a nonlinear plant (excluding saturation) the set of linear system functions with reduced sensitivity to parameter change may be attained over the significant system frequency range. However, with a nonlinear plant, the crucial difficulty of insuring system stability remains a nonlinear problem and is outside the scope of this paper.

The Role of System Configuration in Achieving Design Objectives

This section considers to what extent the design objectives are achievable and the role of system configuration in obtaining them. It is assumed for the moment that the plant is accessible only at its input and output points. There are therefore only three sets of variables,

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[1] H. Freeman, "A synthesis method for multipole control systems," *Trans. AIEE*, pt. II, vol. 76, pp. 28–31; March, 1957.

[2] A. S. Boksenboom and R. Hood, "General Algebraic Method Applied to Control Analysis of Complex Engine Types," National Advisory Committee for Aeronautics, Washington, D. C., Tech. Rept. No. 980; April, 1949.

[3] R. J. Kavanagh, "Noninteraction in linear multivariable systems," *Trans. AIEE (Applications and Industry)*, vol. 76, pp. 95–100; May, 1957.

[4] H. Freeman, "Stability and physical reliability considerations in the synthesis of multipole control systems," *Trans. AIEE (Applications and Industry)*, vol. 77, pp. 1–5; March, 1958.

each set represented by a node in Fig. 1(a). R is an $n \times 1$ matrix whose elements are the Laplace transforms of the input command signals, X is an $n \times 1$ matrix of the plant input signals. P is the $m \times n$ matrix of plant transfer functions, and C is the $m \times 1$ matrix of output signal transforms.

Since no signal is allowed to reach node C except through P, the system response is entirely determined by the values of X. Also, X must be determined by the signals it receives from R and C. There are accordingly only two essential sets of transfer functions, one from R to X and the other from C to X [see Fig. 1(b)]. Therefore, as far as the design objectives are concerned, Fig. 1(b) is sufficiently general and no other configuration can do any more. For example, Fig. 1(c) appears to possess more degrees of freedom, but it is reducible to Fig. 1(b), with $G=G_1G_2G_3$, $H=H_3+G_3H_2+G_2G_3H_1$, and the effect on the design objectives is precisely the same. This means that there are two degrees of freedom which, it will be shown, may be used to determine a matrix of system transfer functions $T(C=TR)$ and, independently, either a set of sensitivities of these transfer functions to variations in plant parameters or a matrix of system responses to external disturbances entering the plant. When there is access to more variables within the plant, *i.e.*, when the designer may feed back from points other than the output terminals, the above conclusions are not seriously modified. It is then possible to control independently the sensitivities of T to various portions of the plant or to control separately the system response to disturbances entering at different points in the plant.

It is interesting to inquire whether there may be substantial differences between various configurations in such matters as system sensitivity to the compensating networks and system response both to noise entering elsewhere than via the plant and to the gain-bandwidth requirements of the compensating networks. Since it does not appear possible to generalize, a numerical example for a single-variable plant is considered. The six different configurations shown in Fig. 2 are compared with respect to (a) the gain-bandwidth demands on the compensating networks, (b) the sensitivity of the system to the compensating networks, (c) the level of the output at various points for noise entering with the useful signal, and (d) the noise entering through the feedback path. For a fair comparison all configurations have the same loop transmission in order that they may all be the same with respect to the major design objectives. This means that they all have the same loop transmission L, and the same $T=C/R$, at least over the significant frequency range. The example chosen (Fig. 3)

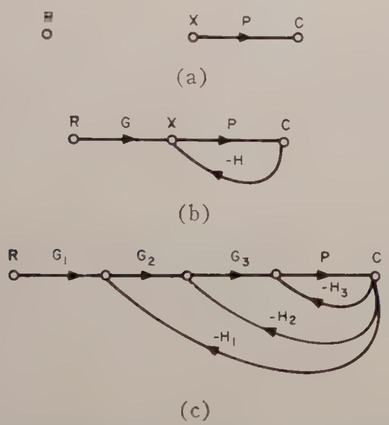

Fig. 1—(a) The three essential variables. (b) A sufficiently general configuration. (c) Alternate and more complex configuration.

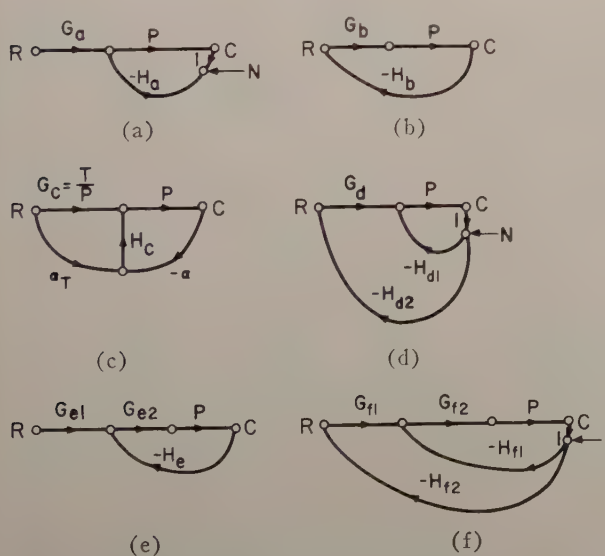

Fig. 2—Six alternate configurations.

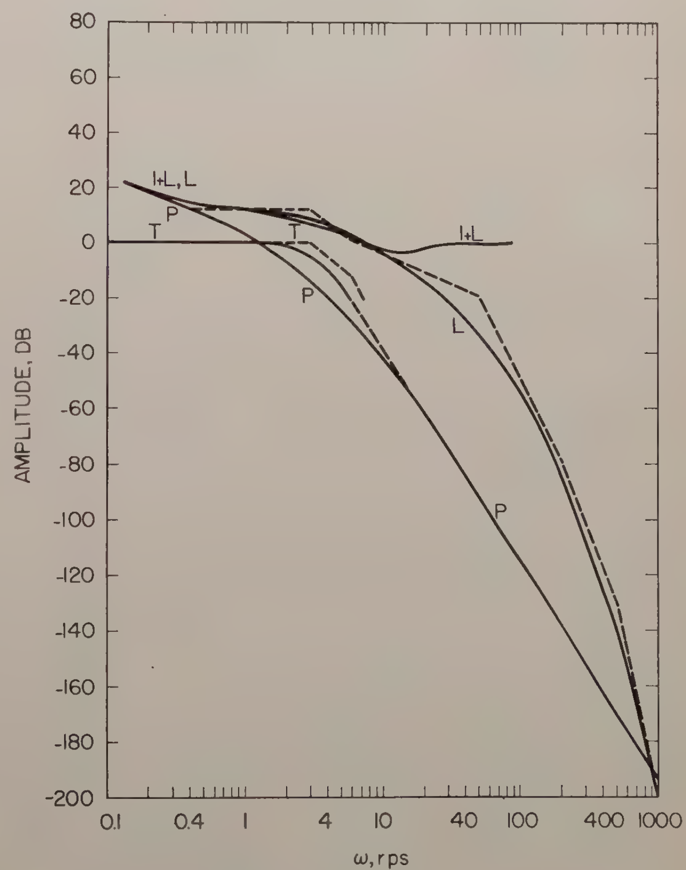

Fig. 3—Example of single-variable system.

TABLE I

TABLE OF G AND H FUNCTIONS AND SIGNAL LEVELS FOR CONFIGURATIONS SHOWN IN FIG. 2

	A	B	C	D	E	F
H	$H_a = \frac{L}{P}$	$H_b = \frac{L}{T(1+L)}$	$\alpha H_c = \frac{L}{P}$	$H_{d1} = \frac{L}{P} - H_{d2}G_d$	$\frac{L}{PG_{e2}}$	$H_{f1} \sim 1$ $H_{f2} \sim 1$
G	$G_a = \frac{T(1+L)}{P}$	$G_b = \frac{T(1+L)}{P}$	$G_c = \frac{T}{P}$	$G_d = \frac{T(1+L)}{P}$	$G_{e1}G_{e2} = \frac{T(1+L)}{P}$	$G_{f1}: \frac{T(1+L)}{L}$ $G_{f2}: \frac{L}{P}$
Signal level for $R=1$ at output of						
G	$\frac{T(1+L)}{P}$	$\frac{T}{P}$	$\frac{T}{P}$	$\frac{T(1+L)}{P}$	$G_{e1}: G_{e1}$ $G_{e2}: \frac{T}{P}$	$G_{f1}: \frac{T}{L}$ $G_{f2}: \frac{T}{P}$
H	$\frac{LT}{P}$	$\frac{L}{1+L}$	—	$H_{d1}: \frac{LT}{P}$ $H_{d2}: TH_{d2}$	$\frac{TL}{PG_{e2}}$	$H_{f1}: TH_{f1} \quad T$ $H_{f2}: TH_{f2} < T$
Signal level for $N=1$ at output of						
G	—	$\frac{L}{P(1+L)}$	—	$\frac{T}{P} H_{d2}$	G_{e1}: — $G_{e2}: \frac{L}{P(1+L)}$	$G_{f1}: \frac{H_{f2}T}{L} < \frac{T}{L}$ $G_{f2}: \frac{L}{P(1+L)}$
H	$\frac{L}{P(1+L)}$	$\frac{L}{T(1+L)^2}$	$\frac{L}{P(1+L)}$	$H_{d1}: \frac{L}{P(1+L)}$ $H_{d2}: \frac{H_{d2}}{1+L}$	$\frac{L}{P(1+L)G_{e2}}$	$H_{f1}: \frac{H_{f1}}{1+L} \sim \frac{1}{1+L}$ $H_{f2}: \frac{H_{f2}}{1+L} < \frac{1}{1+L}$

corresponds to the lead type of compensation since the system-response bandwidth is greater than that of the plant. The bandwidth over which there is negative feedback (with all its advantages) is about two and one-half times the system-response bandwidth, and the gain and phase margins are fairly conservative. It is assumed that the magnitude of the system response $T = C/R$ is prescribed up to the frequency (5.7 rps) at which it decreases to -20 db. In this example, the plant is driven hard to give a response faster than its own natural inclinations and appreciable loop gain L (*i.e.*, the benefit of negative feedback) is desired over a much larger bandwidth than the plant is capable of giving. It is not surprising, therefore, that the compensating networks must have large gains over a frequency range considerably greater than that of the system itself. The example thus represents one of the more exacting types of design.

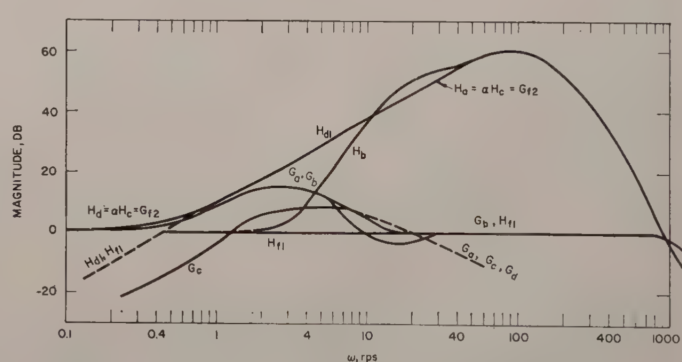

Fig. 4—Characteristics of the configurations shown in Fig. 2.

Can any significant conclusions be drawn, at least for this kind of design example? G and H functions for the various configurations in Fig. 2 in terms of T and L are given in Table I, and Bode plots of the various G and H transfer functions are drawn in Fig. 4. In some cases, a compensating network (a G or an H) must make up the difference between the gain-bandwidth demands of the loop transmission L and the amount supplied by the plant P. In other cases, a compensating network must supply the difference between the requirements of both T and L and the amount contributed by P.

The various configurations permit different divisions between the G's and H's. Configuration A has an advan-

tage over B because T enters only in the expression for G_a and therefore T can be brought down very fast to ease the demand on G_a. This cannot be done with the B structure. In the latter, both G and H must be under control until L becomes less than P; in this example this occurs more than two decades beyond the system bandwidth. The C structure (called conditional feedback[5] or model feedback) is substantially the same as A. In structure D, H_{d1} is fairly independent of T and therefore this structure has the same advantage over B that A and C have; *i.e.*, only one function, H_{d1}, need be controlled over a very large frequency range. Structures E and F represent modifications of A and D, respectively. There is some advantage in B, D, F over A, E: up to approximately 15 rps, the feedback is negative and therefore over this limited frequency range the G elements in B, D, F benefit. There is no particular advantage in E over A in the gain-bandwidth demands but there may be an advantage in F over D. In F,

$$G_{f1} = \frac{T(1+L)}{PG_{f2}} \tag{1}$$

$$H_{f1} = \frac{L}{PG_{f2}} - H_{f2}G_{f1}. \tag{2}$$

If G_{f2} is made equal to L/P, then

$$G_{f1} = \frac{T(1+L)}{L} \tag{3}$$

can probably be passive but $H_{f1} \doteq 1$ must be controlled over a wide frequency range (until L is equal to P). In return for the latter, the requirements on G_{f1} are very slight. There may also be some advantage in E and F in terms of the required dynamic range of the G's and H's, especially if high-frequency noise is a problem. (The noise output of the over-all system is exactly the same for all the configurations for noise that enters with the signal or that enters anywhere in the plant, or at N in Fig. 2 via the feedback loop.) The signal level at the output of G_a in Fig. 2(a) is, of course, G_a with its peaked high-frequency response. In Fig. 2(e) it can be reduced by letting G_{e2} be larger, without affecting the level at the output of G_{e2} which is fixed at T/P. There is a similar saving in the signal level at the output of H_e. However, each of the high-gain elements in each configuration is very susceptible to high-frequency noise, entering by the feedback loop via N. It would seem from the above that there may be some advantage in configurations with more than the minimum two degrees of freedom, and that these advantages are in the gain-bandwidth and in the dynamic range requirements of the compensating networks. In an example such as the above, the difference in system sensitivity to the compensating networks is minor because the demand on these networks is chiefly over that frequency range in which the loop transmission is less than unity.

[5] G. Lang and J. M. Ham, "Conditional feedback systems—a new approach to feedback control," *Trans. AIEE* (*Applications and Industry*), vol. 74, pp. 152–161; July, 1955.

Steps in the Design of a Multivariable Control System

There are three principal steps in the synthesis of a multivariable control system.

1) Realization of the desired set of system transfer functions (the matrix T) over their significant frequency range (roughly an octave or two beyond their half-power points, unless there are special reasons for going beyond). It is not a good idea to try *a priori* to specify the behavior of the elements of T beyond this range, as will be seen when step 3 is considered.

2) Next, the magnitude of the loop transmissions over the significant frequency ranges must be fixed in order to obtain the desired insensitivity to plant parameter variation and/or the desired rejection of disturbing signals.

3) Finally, the loop-transmission functions should be allowed to decrease in magnitude as fast as is consistent with the stability problem in the face of the plant parameter variations. This involves design effort well outside the useful system frequency range. The practical aspects of design are best handled at this point. Thus, it is unrealistic to specify building blocks whose transfer functions have more or even as many zeros as poles. Nevertheless, such functions usually result from the first two steps. In step 3, the additional poles may be introduced as soon as feasible without affecting system stability. That is why the behavior of the T elements in this far-off frequency range is best left to this part of the design. The final result is a design without the ambiguities of building blocks with transfer functions that are infinite or even finite at infinite frequency. These three steps will now be considered in detail.

Step 1—Realization of T

The plant has n inputs and m outputs. If $n<m$, $m-n$ outputs are dependent functions of the others and may be ignored, leaving a plant described by an $n\times n$ matrix P. If $m<n$, $n-m$ fictitious outputs are invented, precisely equal to $n-m$ inputs, so that again there results an $n\times n$ matrix P that describes the plant. For example, if $n=4$, $m=2$, the plant matrix is

$$P = \begin{bmatrix} p_{11} & p_{12} & p_{13} & p_{14} \\ p_{21} & p_{22} & p_{23} & p_{24} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

The above procedure is the same as that described in the literature[1–4] except that in this paper P will not be shown in partitioned form. The configuration used in the development is that shown in Fig. 2(b); it is redrawn in Fig. 5. If any other configuration should be preferred, the values of the new building blocks can be obtained from G, P, and H.

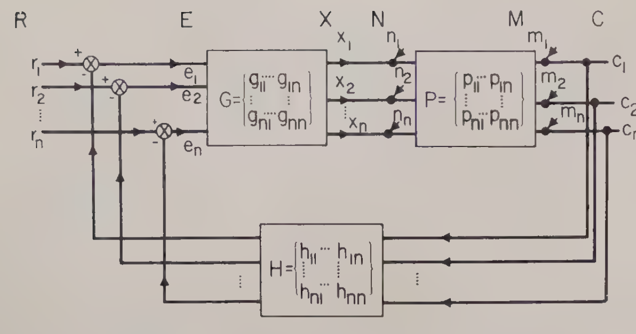

Fig. 5—Multivariable feedback control system configuration used in developing the synthesis procedure.

Capital letters are used for matrices, of which there are two kinds: $n \times n$ matrices, such as T, P, H, G; and $n \times 1$ matrices representing transforms of signals, such as R, E, X, N, M, C. The matrix elements are designated by small letters with pairs of subscripts to denote their position in the matrix, *e.g.*, p_{ij}, h_{ij}, t_{ij}, g_{ij}. Capital letters with pairs of subscripts denote cofactors, *e.g.*, P_{ij} is the ijth cofactor of P. The determinant of an $n \times n$ matrix is indicated by the Δ notation; *e.g.*, Δ_p, Δ_g are the determinants of P and G, respectively, while Δ_P is the determinant of the matrix of the cofactors of P.

In Fig. 5, with the external disturbances $N = M = 0$,

$$C = PX = PGE = PG(R - HC); \tag{4}$$

therefore

$$C = (I + PGH)^{-1}PGR = TR, \tag{5}$$

from which

$$H = T^{-1} - (PG)^{-1}. \tag{6}$$

Or one may write

$$E = R - HPGE, \tag{7}$$

and therefore

$$\begin{aligned} C &= PGE \\ &= PG(I + HPG)^{-1}R \\ &= TR, \end{aligned} \tag{8}$$

i.e.,

$$\begin{aligned} T &= (I + PGH)^{-1}PG \\ &= PG(I + HPG)^{-1}. \end{aligned} \tag{9}$$

Eq. (9) is used to find H in the significant frequency range of T, as noted in step 1. It can be applied as soon as G is known, because T and P are known *a priori*. G is obtained from the desired insensitivity of the T elements to variations in the parameters of P and/or the desired response to the external disturbances, N and M.

Step 2—Effect of Variations of Plant Parameters

T is the nominal transmission matrix when the plant has the nominal matrix P. When the latter has a different set of values P', the resulting new transmission matrix is T', *i.e.*,

$$P' = P + \Delta P \tag{10a}$$

$$T' = T + \Delta T. \tag{10b}$$

Since

$$T = PG(I + HPG)^{-1}, \tag{11a}$$

$$T' = (I + P'GH)^{-1}P'G, \tag{11b}$$

and from (9) and (10),

$$\begin{aligned} T'^{-1}\Delta T = T'^{-1}(T' - T) &= G^{-1}P'^{-1}(P'GH + I) \\ &\quad \cdot [(P'GH + I)^{-1}P'G - (PGH + I)^{-1}PG] \\ &= G^{-1}P'^{-1}\{P'G - [(P + \Delta P)GH + I] \\ &\quad \cdot [(PGH + I)^{-1}PG]\} \\ &= G^{-1}P'^{-1}(\Delta P)G(I - HT). \end{aligned} \tag{12}$$

With the aid of (6), the latter becomes

$$T'^{-1}\Delta T = G^{-1}P'^{-1}(\Delta P)P^{-1}T \triangleq S$$

($\triangleq$ means "equal by definition"). (13)

Let the elements in ΔP be designated by δ_{ij} and those in ΔT by τ_{ij}, and multiply out both sides of (13). The result is (recall $P_{\alpha\beta}$ is the cofactor of the $\alpha\beta$ element in the P matrix, etc., Δ_p is the determinant of P, Δ_P is the determinant of the cofactors of P, etc.),

$$S = G^{-1}P'^{-1}(\Delta P)P^{-1}T$$

$$= \begin{bmatrix} \sum P_{yi}'G_{i1}\delta_{yj}P_{\nu j}t_{\nu 1} & \sum P_{yi}'G_{i1}\delta_{yj}P_{\nu j}t_{\nu 2} \cdots & \sum P_{yi}'G_{i1}\delta_{yj}P_{\nu j}t_{\nu n} \\ \sum P_{yi}'G_{i2}\delta_{yj}P_{\nu j}t_{\nu 1} & \cdots & \sum P_{yi}'G_{i2}\delta_{yj}P_{\nu j}t_{\nu n} \\ \vdots & & \vdots \\ \sum P_{yi}'G_{in}\delta_{yj}P_{\nu j}t_{\nu 1} & \cdots & \sum P_{yi}'G_{in}\delta_{yj}P_{\nu j}t_{\nu n} \end{bmatrix} \frac{1}{\Delta_g\Delta_p\Delta_{p'}}$$

$$= \frac{\begin{bmatrix} \sum T_{i1}'\tau_{i1} & \sum T_{i1}'\tau_{i2} \cdots & \sum T_{i1}'\tau_{in} \\ \sum T_{i2}'\tau_{i1} & \cdots & \sum T_{i2}'\tau_{in} \\ \vdots & & \\ \sum T_{in}'\tau_{i1} & \cdots & \sum T_{in}'\tau_{in} \end{bmatrix}}{\Delta_{t'}} = T'^{-1}(\Delta T). \tag{14}$$

The set of n τ's denoted by τ_{j1} is obtained from the n simultaneous equations derived by equating the n elements in the first column of (14); *i.e.*,

$$\Delta_{t'} s_{i1} = \sum_j T_{j1}'\tau_{j1} \qquad \text{for } i = 1, 2, \cdots n. \tag{15}$$

By using Cramer's rule,

$$\tau_{11} = \frac{\Delta_{t'}}{\Delta_{T'}}(s_{11}M_{11} + s_{21}M_{21} + \cdots + s_{n1}M_{n1}), \tag{16}$$

where M_{ij} is the ij cofactor of the $\Delta_{t'}T'^{-1}$ matrix. By Jacobi's theorem,[6]

$$M_{ij} = t_{ji}'\Delta_{t'}^{n-2}, \tag{17}$$

while

$$\Delta_{T'} = \Delta_{t'}^{n-1}. \tag{18}$$

Therefore,

$$\tau_{11} = s_{11}t_{11}' + s_{21}t_{12}' + \cdots + s_{n1}t_{1n}'. \tag{19}$$

In general,

$$\tau_{\alpha\beta} = \sum s_{j\beta}t_{\alpha j}'. \tag{20}$$

From (14) and by letting δ_n^m represent the Kronecker delta,

$$\begin{aligned}\Delta_p\Delta_{p'}\Delta_g s_{\alpha\beta} &= \sum P_{yi}'G_{i\alpha}\delta_{yj}P_{\nu j}t_{\nu\beta}\\ &= \sum P_{yi}'G_{i\alpha}p_{yj}'P_{\nu j}t_{\nu\beta} - \sum P_{yi}'G_{i\alpha}p_{yj}P_{\nu j}t_{\nu\beta}\\ &= \sum \Delta_{p'}\delta_j^iG_{i\alpha}P_{\nu j}t_{\nu\beta} - \sum \Delta_p\delta_\nu^yP_{yi}'G_{i\alpha}t_{\nu\beta}\\ &= \Delta_{p'}\sum G_{i\alpha}P_{\nu i}t_{\nu\beta} - \Delta_p\sum P_{\nu i}'G_{i\alpha}t_{\nu\beta}\\ &= \sum G_{i\alpha}t_{\nu\beta}(\Delta_{p'}P_{\nu i} - \Delta_pP_{\nu i}').\end{aligned} \tag{21}$$

Therefore,

$$\Delta_g\tau_{\alpha\beta} = \sum t_{\alpha j}'G_{ij}t_{\nu\beta}\left(\frac{P_{\nu i}}{\Delta_p} - \frac{P_{\nu i}'}{\Delta_{p'}}\right). \tag{22}$$

For example, if $n=2$,

$$\begin{aligned}\Delta_g\tau_{12} = (t_{11}'G_{11} + t_{12}'G_{12})&\left[t_{12}\left(\frac{P_{11}}{\Delta_p} - \frac{P_{11}'}{\Delta_{p'}}\right)\right.\\ &\left.+t_{22}\left(\frac{P_{21}}{\Delta_p} - \frac{P_{21}'}{\Delta_{p'}}\right)\right]\\ + (t_{11}'G_{21} + t_{12}'G_{22})&\left[t_{12}\left(\frac{P_{12}}{\Delta_p} - \frac{P_{12}'}{\Delta_{p'}}\right)\right.\\ &\left.+ t_{22}\left(\frac{P_{22}}{\Delta_p} - \frac{P_{22}'}{\Delta_{p'}}\right)\right].\end{aligned} \tag{23}$$

The τ's, which represent the arithmetical change in the transmission, may be kept as small as desired by choosing the g's sufficiently large. If over the useful frequency range the τ's are to be kept small,

$$t_{\alpha\beta}' \doteq t_{\alpha\beta}, \tag{24}$$

and (22) is approximated by

$$\Delta_g\tau_{\alpha\beta} \doteq \sum G_{ij}t_{\alpha j}t_{\nu\beta}\left(\frac{P_{\nu i}}{\Delta_p} - \frac{P_{\nu i}'}{\Delta_{p'}}\right). \tag{25}$$

If it is assumed that the τ's and $P_{\nu i}'$'s are known, one may formally solve (25) for the n^2 unknowns G_{ij}/Δ_g, from which the g_{ij} elements are available. Once the latter are known, the H matrix elements are obtainable from (6) and the synthesis in the significant T frequency range is complete.

Only in very exceptional cases would the design effort required above be justified. All the τ's may be kept as small as desired by letting G be a diagonal matrix. In this case (25) becomes

$$\tau_{\alpha\beta} \doteq \sum \frac{t_{\alpha i}t_{\nu\beta}}{g_{ii}}\left(\frac{P_{\nu i}}{\Delta_p} - \frac{P_{\nu i}'}{\Delta_{p'}}\right). \tag{26}$$

It is clear from (26) that it is always possible to choose the g_{ii}'s sufficiently large to make the τ's as small as may be desired.

When rejection of disturbances is important, one is interested in the noise transmission matrix, which from Fig. 5 is

$$T_N = (I + PGH)^{-1}P. \tag{27}$$

From (9) it therefore follows that

$$T = T_NG. \tag{28}$$

G is chosen to obtain the desired noise attenuation. If G is a diagonal matrix,

$$t_{Nij} = \frac{t_{ij}}{g_{jj}}. \tag{29}$$

For noise entering at the output points (Fig. 5),

$$T_M = T(PG)^{-1} \tag{30}$$

and G is chosen to attenuate the noise to the desired extent.

Step 3—Loop Shaping for Stability

The final and usually the most difficult part of the synthesis is to shape the loop transmission in the crossover region to insure system stability despite plant parameter variations. Consider (9) written in the following form:

$$T' = P'G(I + HP'G)^{-1}. \tag{31}$$

P', T' denote the plant and system transmission matrices, respectively, at the new values of the plant parameters. Although it is known that T is stable, the problem is to choose H and G outside the T frequency range such that the system remains stable over the range of P' values. Right half-plane poles of $P'G$ are cancelled out in T'. Right half-plane poles of H appear at worst as zeros in some of the t_{ij}'. Instability is due

[6] H. W. Turnbull, "The Theory of Determinants, Matrices and Invariants," Blackie and Sons, Ltd., London, Eng., 2nd ed., p. 77; 1945.

only to zeros of $|I+P'GH|$ in the right half-plane. Consider $|I+X|$. In general,

$$|I+X| = \Delta_x + \sum_1^n X_{ii} + \sum X_{ii,jj} + \sum X_{ii,jj,kk} + \cdots + \sum X_{ii,jj,\cdots,n-1,n-1} + 1, \tag{32}$$

while

$$|I+P'X| = \Delta_{p'}\Delta_x + \sum P'_{ij}X_{ji} + \sum P'_{\substack{i,j\\ i+1,j+1}} X_{\substack{j,i\\ j+1,i+1}} + \sum P'_{\substack{i,j,k\\ i+1\ j+1,k+1\\ i+2,j+2,k+2}} X_{\substack{k,j,i\\ k+1,j+1,i+1\\ k+2,j+2,i+2}} + \cdots + 1. \tag{33}$$

In the above, X_{ii} is the (ii)th cofactor of X; $X_{ii,jj}$ is the determinant obtained by striking out the i and j rows and columns of X; $P'_{\substack{i,j\\ l,m}}$ is $(-1)^{i+j+l+m}$ times the determinant obtained by striking out of P' rows i, l and columns j, m. Also,

$$X_{ii,ii} = X_{\substack{ij\\ ij}} \triangleq 0. \tag{34}$$

Suppose now that there are r quantities in P' that are variable. The range of values of these r quantities describes a closed region in r-dimensional space. For any one point in this r-dimensional region, the Nyquist sketch of the loop transmission must not encircle the -1 point. If $r=3$, then the closed region is in visualizable space. To explore this region, it is enclosed in a larger but simpler type of region, preferably a solid bounded by planes. To investigate thoroughly the latter region, the eight Nyquist sketches, corresponding to the eight corners of the region, must be sketched. From these sketches it is ascertained whether the system is stable for all points in the three-dimensional region. This method is described in detail in the literature.[7,8] A modification of this procedure, which may be useful, is shown in the examples at the end of this paper. A certain amount of cut and try is inevitable. A guess is made at suitable GH values from which the loop transmissions are sketched. The sketches suggest the changes that must be made should the original trial lead to an unstable system.

Undoubtedly the above procedure entails a great deal of tedious calculation but this appears to be unavoidable. The work can, of course, be done on an analog computer by simulating the n-dimensional system and opening the loop at any single point. For example, if $n=2$ and the loop is opened at p_{11}' (Fig. 6), then the input is applied at point A and the output at point B is

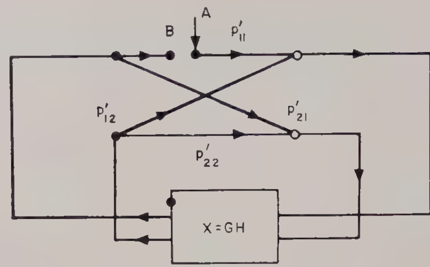

Fig. 6—Loop transmission around p_{11}'.

the loop transmission around p_{11}'. The value of this loop transmission is obtained by expanding (33).

$$|I+P'X| = 1 + p_{22}'x_{22} + p_{21}'x_{12} + p_{12}'x_{21} - p_{12}'p_{21}'\Delta_x + p_{11}'(p_{22}'\Delta_x + x_{11}). \tag{35}$$

The loop transmission around p_{11}' is therefore

$$L_{(p_{11}')} = \frac{-p_{11}'(p_{22}'\Delta_x + x_{11})}{1 + p_{22}'x_{22} + p_{21}'x_{12} + p_{12}'x_{21} - p_{12}'p_{21}'\Delta_x} \tag{36}$$

and is the quantity measured at B. There is no need for a separate simulation of G and H. The elements of $X=GH$ must have their correct design values over the T frequency range and the designer's problem is basically how to bring these values down reasonably fast (in order to economize on gain-bandwidth) without making the system unstable for some value of P' in its region of variation. A reasonable procedure is to assign to X a set of functions and then measure $L_{(p_{11}')}$ for a few frequencies over the range of P'. One soon finds the important crossover frequency range, and by sketching the loci and examining (36) eventually finds a satisfactory set of values for X. After X is known, GH is known, but there is freedom in the individual choice of G and H. This freedom is used to insure that T falls off as fast as may be desired outside its useful band.

One can now appreciate the great increase in the design effort in loop-shaping for stability that must be expended in going from the single-variable control system to even the two-variable system. In loop-shaping to insure stability despite plant parameter variation, the great desideratum is to be able to work with a single loop-transmission curve (even though the loop transmission changes with parameter variation) and shape it to avoid a forbidden region. In the single-variable case (35) becomes

$$1 + p'gh = 1 + \frac{p'}{p} l_0 \tag{37}$$

where $l_0 = pgh$ is the nominal loop transmission. Therefore sketch p/p' for all possible plant variations (no matter how many plant parameters may change) and shape the nominal loop transmission l_0 such that it avoids the locus of p/p'. This is childishly simple compared with the effort involved in picking x's in (35) which are suitable when the range of variations of the primed p's is taken into account.

[7] R. M. Stewart, "A simple graphical method for constructing families of Nyquist diagrams," *J. Aeronaut. Sciences*, vol. 18, pp. 767–768; November, 1951.

[8] F. H. Blecher, "Transistor multiple loop feedback amplifiers," *Proc. Natl. Electronics Conf.*, vol. 13, pp. 19–34; 1957.

Noninteraction

Design labor in steps 1 and 2 is significantly reduced when the specifications demand or permit the principle of noninteraction[2] to be applied. The latter is simply the condition that T is a diagonal matrix. For this case, the sensitivity equation of (25) becomes

$$\Delta_g \tau_{\alpha\alpha} \doteq \sum G_{i\alpha} t_{\alpha\alpha} t_{\beta\beta} \left(\frac{P_{\beta i}}{\Delta_p} - \frac{P_{\beta i}'}{\Delta_{p'}} \right). \tag{38}$$

Again it is possible to keep τ as small as desirable with G a diagonal matrix; in such a case, each $\tau_{\alpha\beta}$ consists simply of

$$\tau_{\alpha\beta} \doteq \frac{t_{\alpha\alpha} t_{\beta\beta}}{g_{\alpha\alpha}} \left(\frac{P_{\beta\alpha}}{\Delta_p} - \frac{P_{\beta\alpha}'}{\Delta_p'} \right). \tag{39}$$

The proper choice of the diagonal G matrix is straightforward. The elements in the H matrix are also easily found from (6):

$$\begin{bmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \vdots & & & \\ h_{n1} & & \cdots & h_{nn} \end{bmatrix} = \begin{bmatrix} \frac{1}{t_1} & 0 & 0 & \cdots \\ 0 & \frac{1}{t_1} & 0 & \\ & & \ddots & \\ 0 & & \cdots & \frac{1}{t_n} \end{bmatrix}$$

$$- \begin{bmatrix} \frac{1}{g_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{g_2} & \cdots & \\ & & \ddots & \\ 0 & & \cdots & \frac{1}{g_n} \end{bmatrix} \begin{bmatrix} P_{11} & P_{21} & \cdots & P_{n1} \\ P_{12} & P_{22} & \cdots & P_{n2} \\ \vdots & & & \\ P_{1n} & & \cdots & P_{nn} \end{bmatrix}. \tag{40}$$

Direct expansion of (40) leads to

$$h_{\alpha\alpha} = \frac{1}{t_{\alpha\alpha}} - \frac{P_{\alpha\alpha}}{g_{\alpha\alpha}\Delta_p} \tag{41}$$

$$h_{\alpha\beta} = \frac{-P_{\beta\alpha}}{g_{\alpha\alpha}\Delta_p}, \qquad \text{for } \alpha \neq \beta. \tag{42}$$

The selection of H and G in the T frequency range is therefore easy. However, if there is substantial plant parameter variation, major design effort is outside the T frequency range and proceeds as before. The matrix $X = GH$ of (33) is easier to calculate, and once its final value has been chosen, it is easier to select appropriate individual values for G and H, but the shaping of the loop transmission remains substantially as difficult as before.

One more simplification may be made. If there is substantial plant parameter variation, the achievement of noninteraction is illusory because, from (42), it is achieved only at one value of the plant parameters. Actually, the off-diagonal terms of T can be made as small as desired in the T frequency range by choosing the elements of G sufficiently large. The off-diagonal transmissions are essentially equivalent to external disturbances in a diagonal type of plant and are reduced in the same manner by large g terms. Therefore one may also make H a diagonal matrix. Essentially what all this amounts to is that in the T frequency range, large loop transmission leads to $T \doteq H^{-1}$, just as in the single-variable system $t \doteq 1/h$. It is important to note that the pure synthesis effort is only in the T frequency range. Loop-gain shaping for stability in spite of plant parameter variations remains a major problem despite the fact that with both G and H diagonal matrices, $X = GH$ is diagonal and the loop-transmission calculation is simplified. Thus for $n = 2$, (35) becomes

$$|I + P'X| = 1 + p_{22}'x_2 + p_{11}'x_1 + \Delta_{p'}x_1x_2. \tag{43a}$$

There is one special case worth mentioning. Suppose that either p_{12} or p_{21} is zero; then (43a) becomes

$$|I + P'X| = (1 + p_{22}'x_2)(1 + p_{11}'x_1) \tag{43b}$$

and the stability problem is tremendously simplified.

When feedback is used primarily to reject disturbing signals and there is little plant parameter variation, it is obvious that the problem of loop shaping to insure stability is considerably reduced.

Example 1

Specifications:

$$P = \begin{bmatrix} \dfrac{+a}{\tau\left(s + \dfrac{1}{\tau}\right)} & \dfrac{b}{\tau\left(s + \dfrac{1}{\tau}\right)} \\ 0 & c \end{bmatrix}. \tag{44}$$

The quantities a, b, c, τ can each have a value anywhere from one to four. With regard to T, t_{11} and t_{22} are to have bandwidths of $\frac{1}{2}$ and 1 rps, respectively. The values of t_{11} and t_{22} are not to change by more than 10 per cent from dc up to their half-power points. Also t_{12} and t_{21} are to be each less than 0.05 over the useful frequency range.

Design: The magnitude of the g's in the useful frequency range required to maintain the desired insensitivity is first ascertained. From (39),

$$\frac{g_{11}\tau_{11}}{t_{11}^2} = \left(\frac{1 + s\tau}{a} - \frac{1 + s\tau'}{a'} \right) \tag{45a}$$

$$\frac{g_{22}\tau_{22}}{t_{22}^2} = \left(\frac{1}{c} - \frac{1}{c'} \right) \tag{45b}$$

$$\frac{g_{11}\tau_{12}}{t_{22}^2} = \left(\frac{b}{ac} - \frac{b'}{a'c'} \right) \tag{45c}$$

$$t_{21} = \tau_{21} = 0. \tag{45d}$$

Suppose that the nominal values of the parameters are taken to be $a=b=c=1$, $\tau=4$. Then the maximum value of (45b) is $\frac{3}{4}$, while the extremes of (45c) are $-15/16$ and 3. Therefore $g_{22} \geq 7.5$ up to approximately 1 rps, and g_{11} must be ≥ 60 ups to $\frac{1}{2}$ rps. This value of g_{11} is more than enough to satisfy the specifications on τ_{11}.

The next step is to shape the loop transmission so that the system remains stable despite the known extremes of plant parameter variation. In this example, (43b) applies, and

$$(1 + x_2c')\left(1 + \frac{x_1a'}{\tau's + 1}\right)$$

must have no right half-plane zeros. Since $x_2 = g_{22}h_{22}$, $x_1 = g_{11}h_{11}$, it is desirable that each x have at least two poles and that it decrease with frequency as rapidly as possible to make the demands on the g's and h's as easy as possible. This decrease should be accomplished subject, of course, to the requirements on the g's in the useful frequency range and in order that the zeros of $|I+P'X|$ should not be too much underdamped. It is easy to pick a simple x_2 with two poles, *viz.*,

$$x_2 = \frac{(7.5)(25)}{(s+1)(s+25)}$$

such that in the worst case, when $c'=4$, the damping factor of the zeros of $1+x_2c'$ is approximately $\frac{1}{2}$ and in the useful frequency range $g_2 \approx 7.5$ because $h_2 \doteq 1$ in this range.

To find a suitable x_1, consider the zeros of

$$\left(\frac{\tau's+1}{a'} + x_1\right) = f_1 + f_2. \tag{46}$$

It is necessary that the locus of x_1 be properly shaped with respect to the loci of

$$-f_1 = -\left(\frac{\tau's+1}{a'}\right).$$

For example, Fig. 7(a) represents a stable system and Fig. 7(b) represents an unstable system; Fig. 8(a) is stable and Fig. 8(b) is conditionally stable. In our specific problem from the point of view of stability we are being conservative if we approximate

$$f_1 = \frac{\tau's+1}{a'}$$

by $(\tau'/a')s = hs$ with h a number that may have any value between 1 and 4. To guard against conditional stability and to obtain a locus like that in Fig. 7(a), the phase lag of x_1 should be less than 90° so long as $|x_1| > |f_1|$. With a little experimentation, a conservative value of x_1 is found to be

$$x_1 = g_{11}h_{11} = \frac{4000(s+1.5)^2}{\left(s+\frac{1}{2}\right)^2(s+20)(s+30)}.$$

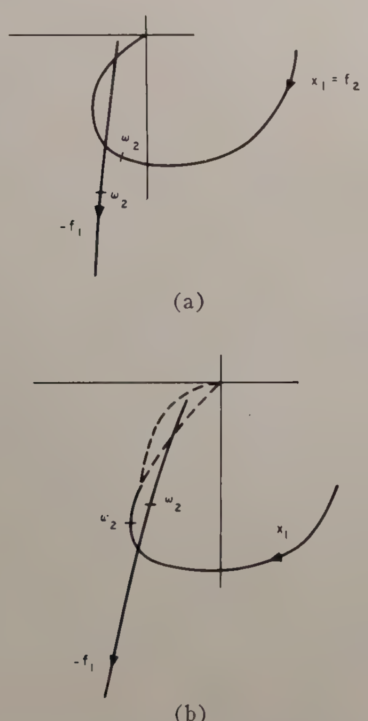

Fig. 7—Nyquist diagrams. (a) Stable, (b) unstable.

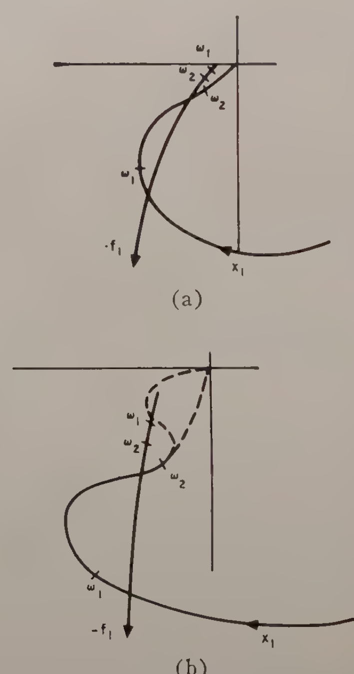

Fig. 8—Nyquist diagrams. (a) Stable, (b) conditionally stable

Now that $x_1=g_{11}h_{11}$, $x_2=g_{22}h_{22}$ are known, (41) is used to obtain

$$g_{11}h_{11} = \frac{g_{11}}{t_{11}} - 4\left(s + \frac{1}{4}\right) \tag{47a}$$

$$g_{22}h_{22} = \frac{g_{22}}{t_{22}} - 1. \tag{47b}$$

If we pick

$$t_{22} = \frac{212.5}{25a} \frac{(s+a)(s+25)}{(s+1)(s^2+26s+212.5)}, \tag{48a}$$

in order to simplify the design of g_{22} and h_{22}, then

$$g_{22} = \frac{8.5(s+a)}{a(s+1)^2} \tag{48b}$$

$$h_{22} = \frac{(s+1)(22.5a)}{(s+a)(s+25)} \tag{48c}$$

The value of a may be a compromise between the desire to bring h_{22} down as fast as possible and the requirements on t_{22}.

From (47a) and the value of x_1

$$\frac{g_{11}}{t_{11}} = \frac{4(s+1.1)(s^2+1.8s+4)(s+16)}{(s+0.5)^2(s+20)}. \tag{49}$$

If t_{11} is taken to be

$$t_{11} = \frac{0.1(s+20)}{(s+\frac{1}{2})(s^2+1.8s+4)}, \tag{50a}$$

then

$$g_{11} = \frac{0.4(s+1.1)(s+16)}{(s+\frac{1}{2})^3} \tag{50b}$$

and

$$h_{11} = \frac{10{,}000(s+0.5)(s+1.5)^2}{(s+1.1)(s+16)(s+20)(s+30)}. \tag{50c}$$

The synthesis is complete and there is no ambiguity in the g's and h's since they all have more poles than zeros. When the above values are substituted into (9), the results check with the above and t_{12} satisfies the specifications.

The design effort in the above example was not great, thanks to the fact that p_{21} was zero.

Example 2

Specifications: A plant with three inputs and two outputs. The plant matrix is

$$P = \frac{1}{\tau s+1}\begin{bmatrix} a_1 & 0.5a_2 & 1 \\ 0.5a_1 & a_2 & 0.5 \\ 0 & 0 & \tau s+1 \end{bmatrix}, \tag{51a}$$

from which the matrix of the cofactors is found to be

$$\frac{1}{(\tau s+1)^2}\begin{bmatrix} a_2 & -0.5a_1 & 0 \\ -0.5a_2 & a_1 & 0 \\ -0.75a_2 & 0 & 0.75a_1a_2 \end{bmatrix}. \tag{51b}$$

The parameters τ, a_1, a_2 may each have values anywhere from one to four. The bandwidth of t_{11} and t_{22} is to be each 1 rps and the values of t_{11} and t_{22} are to vary by not more than 10 per cent in this range due to plant parameter variation. The cross transmissions t_{12}, t_{21}, etc., are to be less than 0.05 over the 1-rps bandwidth.

Design: Because of the significant parameter variations there is no point in adjusting for noninteraction; hence diagonal G and H are used. With the aid of (39) and with $t_{11}=t_{22}$, the variations in the transmissions are found to be

$$\tau_{31} = \tau_{32} = \tau_{23} = 0 \tag{52}$$

$$\tau_{11} = \frac{4}{3}\frac{t_{11}^2}{g_{11}}\left(\frac{\tau s+1}{a_1} - \frac{\tau' s+1}{a_1'}\right) = -2\tau_{12} \tag{53}$$

$$\tau_{21} = \frac{2}{3}\frac{t_{11}^2}{g_{22}}\left(\frac{\tau' s+1}{a_2'} - \frac{\tau s+1}{a_2}\right) = -\tfrac{1}{2}\tau_{22} \tag{54}$$

$$\tau_{13} = \frac{t_{11}t_{33}}{g_{11}}\left(\frac{\tau' s+1}{a_1'} - \frac{\tau s+1}{a_1}\right). \tag{55}$$

The nominal values of the plant are taken as $a_1=a_2=2$, $\tau=4$; t_{33} is taken as unity and since $p_{33}=1$, a constant, there is no need for feedback around this variable; hence $h_{33}=0$ and $g_{33}=1$. The specifications on the τ's are satisfied if g_{11} and g_{22} are each 30 over the passband. We can also take $t_{11}=t_{22}=t$, $g_{11}=g_{22}=g$, $h_{11}=h_{22}=h$; therefore $x_{11}=g_{11}h_{11}=x_2=x$, and , of course, $x_3=0$.

The stability problem is greatly simplified by the fact that $x_3=0$. The polynomial (33) that must be examined for no right half-plane zeros is

$$1 + \frac{0.75a_1'a_2'x_1x_2}{(\tau' s+1)^2} + \frac{a_1'x_1}{\tau' s+1} + \frac{a_2'x_2}{\tau' s+1}.$$

This is rewritten as

$$\left(\frac{\tau' s+1}{a_1'a_2'}\right)[\tau' s + 1 + (a_1' + a_2')x] + 0.75x^2$$

$$= f_1 + f_2. \tag{56}$$

The discussion regarding Figs. 7 and 8 applies, except that the problem here is more difficult because f_1 is a function of f_2. But with some preliminary thinking, the specific loop-shaping required is apparent. The function $x=gh$ should have at least two more poles than zeros to permit a realistic g and a realistic h. Therefore, assuming lead compensation at low frequencies is not needed, $0.75x^2=f_2$ will have the frequency locus indicated (qualitatively, of course) in Fig. 9. At low frequencies

$$-f_1 \doteq -\left(\frac{\tau's+1}{a_1'a_2'}\right)(a_1'+a_2')\,x$$

and it would therefore be like the segments M_1 or M_2 in Fig. 9. At high frequencies

$$-f_1 \doteq \frac{-(\tau's+1)^2}{a_1'a_2}$$

and therefore $-f_1$ tends toward the positive real axis, as shown in the figure. In order to have the situation shown in Fig. 8(a) (unconditionally stable), it is necessary to decrease the magnitude of f_2 without too much phase shift (this is always the problem in loop-transmission shaping).

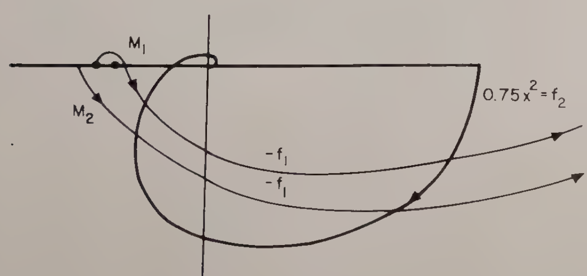

Fig. 9—Preliminary construction for requirements on x.

this is sketched in Fig. 10. The resulting $-f_1$ and f_2 are sketched in Fig. 11 for $\tau=4$, $a_1=a_2=4$; $\tau=1$, $a_1=a_2=1$; $\tau=1$, $a_1=a_2=4$ and for $\tau=4$, $a_1=a_2=1$. Consider the loci of $-f_{1a}$, $-f_{1c}$. As τ changes in value from 1 to 4 (with $a_1=a_2=4$), the point corresponding to $\omega=4$ describes a locus starting from A and terminating in B (see Fig. 11). For the system to be stable, this locus must pass to the left of C, which is the point of $\omega=4$ on the f_2 locus. Similar loci between corresponding points on the $-f_1$ curves must be considered. It appears that the design is very likely stable but would, in any case, result in system poles which at some plant parameter values tend to be too much underdamped.

It is clear what should be done to remedy the matter. The design for x_1 must be more conservative with less phase lag for the same decrease in magnitude (with consequent need to control x over a larger frequency range). The final try uses (see Fig. 12)

$$x=\frac{(16)^3(32)(s+2)^2(s+16)^2(s+128)^2}{(s+1)^2(s+8)^2(s+64)^2(s+512)^2} \tag{58}$$

which is probably more conservative than is necessary. The resulting $-f_1$ and f_2 sketches appear in Fig. 13. Clearly, the system is stable for all permissible values of the plant parameter.

Now that $x=gh$ is known, (41) is used to find g/t; *i.e.*,

$$\frac{g_{\alpha\alpha}}{t_{\alpha\alpha}}=g_{\alpha\alpha}h_{\alpha\alpha}+\frac{P_{\alpha\alpha}}{\Delta_p}=x+\frac{4}{3a}=x+\frac{2}{3}. \tag{59}$$

Two near zeros of $g_{\alpha\alpha}/t_{\alpha\alpha}$ are quickly located by approximating x by $8(s+2)^2/(s+1)^2$. The second approximation for x is

$$\frac{2(s+2)^2(s+16)^2}{(s+1)^2(s+8)^2},$$

and with the aid of the first approximation, the resulting factors are found. The process is continued until finally,

$$g=\frac{\frac{2}{3}t(s^2+3.86s+3.80)(s^2+28.4s+216)(s^2+188s+10{,}150)(s^2+950s+389{,}000)}{(s+1)^2(s+8)^2(s+64)^2(s+512)^2}. \tag{60}$$

If the choice for t is taken as

$$t=\frac{(s+a)^2(s+64)^2(s+512)^2\,\dfrac{793}{a^2}}{(s+1)(s^2+28.4s+216)(s^2+188s+10{,}150)(s^2+950s+389{,}000)}, \tag{61}$$

Because f_1 depends on the choice of f_2, some trial and error is unavoidable. The cut and try is greatly facilitated by rapid polar sketches obtained from Bode diagrams of the asymptotic functions. These Bode diagrams are precisely drawn here for illustrative purposes but in practice rapid asymptotic sketches suffice. In the first try, a choice was made with

$$x=\frac{3150(s+4)^2}{(s+1)^2(s+40)^2}; \tag{57}$$

then

$$g=\left(\frac{2}{3}\right)\left(\frac{793}{a^2}\right)\frac{(s+a)^2(s^2+3.86s+3.80)}{(s+1)^3(s+8)^2} \tag{62}$$

and

$$h=\frac{248a^2(s+1)(s+2)^2(s+16)^2(s+128)^2}{(s^2+3.86s+3.80)(s+a)^2(s+64)^2(s+512)^2}. \tag{63}$$

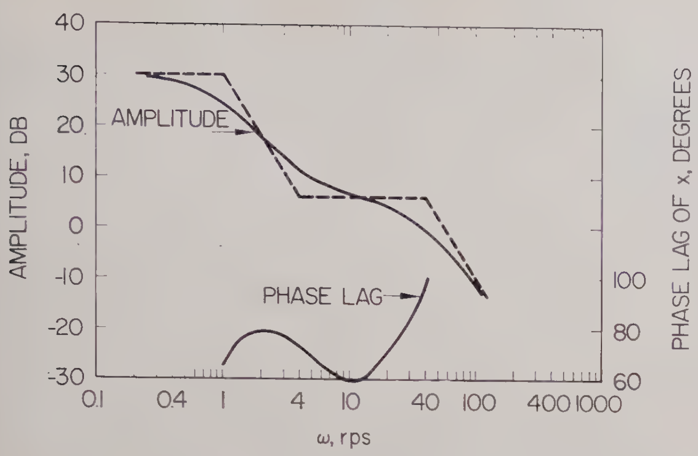

Fig. 10—Design example no. 2 showing first trial value of x.

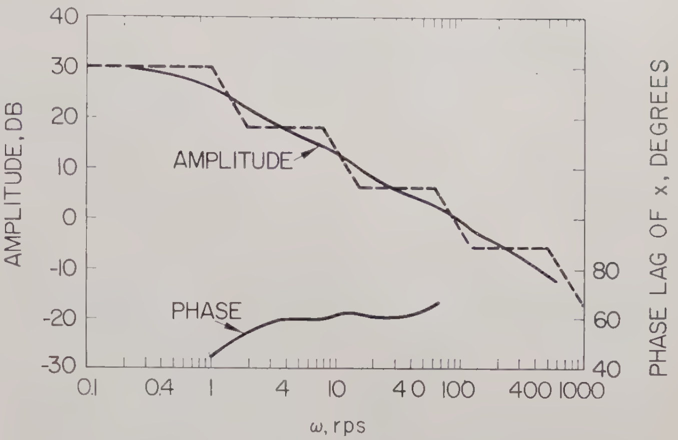

Fig. 12—Design example no. 2 showing second trial value of x.

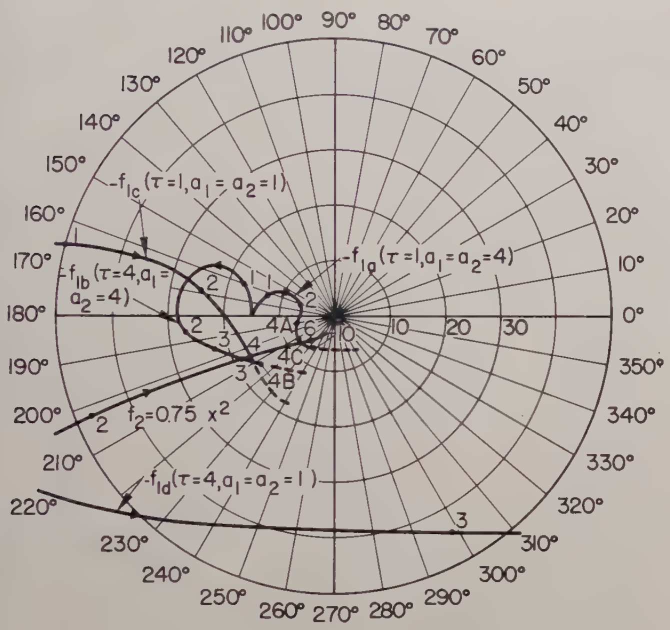

Fig. 11—Sketches of $-f_1$, f_2 for determining stability (example no. 2, trial no. 1).

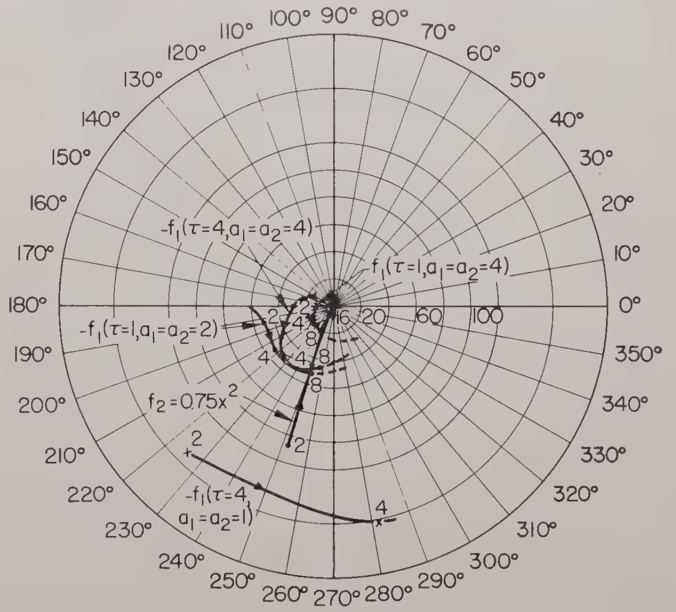

Fig. 13—Sketches of $-f_1$, f_2 for determining stability (example no. 2, trial no. 2).

Conclusion

A synthesis procedure has been presented for the design of linear multivariable feedback control systems to achieve simultaneously:

1) a desired matrix of system transfer functions,
2) a desired insensitivity of these transfer functions to plant parameter variations.

The procedure is applicable to systems with large parameter variations. No constraints such as noninteraction or diagonalization of the loop-transfer function matrix have been assumed. There is, however, some freedom in the manner in which the loop-transmission matrix elements may be chosen to achieve the desired sensitivity reduction. This freedom may be used, as it was used in the numerical examples, to simplify the design calculations, or it may be used for practical considerations, such as to minimize saturation possibilities. One of the really difficult problems, when there is large parameter variation, is to secure stability despite the parameter variation. Some initial cut and try seems unavoidable, but convergence upon a final satisfactory solution is rapid.

Automatic Control of Three-Dimensional Vector Quantities—Part 3*

A. S. LANGE†

Summary—**This is the final part of a three-part paper on the application of vector techniques to automatic control systems whose input, output, and disturbance quantities may be characterized by three-dimensional vectors. In Parts 1 and 2, position and angular velocity vectors were introduced to demonstrate the solution of coordinate conversion and geometric stabilization problems, respectively. In brief, the subject of Part 3 is Newton's Second Law, with the application of the previously defined vector algebra to problems in kinetics. In particular, the behavior of "Newtonian sensors" such as gyroscopes and accelerometers is considered in detail, in order to develop the basic equations which describe the dynamic performance of these devices, determine the errors associated with their usage, and demonstrate the application of the vector algebra to more complex systems.**

IX. Gyroscopic Instruments[37,38]

An important class of problems in applied mechanics is that which involves gyroscopic phenomena. The gyroscopic principle has many useful applications in automatic control systems wherever angular motion is to be measured. The basic principle of gyroscopic behavior (Newton's Second Law) is well known; however, the detailed application of this principle to problems in three dimensions is often avoided because of the complexity involved in accounting for the many terms which arise from inertia and kinematic coupling. It is therefore of interest to make a somewhat detailed analysis of a gyroscopic instrument, as an example of a typical problem in kinetics. This example demonstrates the behavior of an important gyroscopic instrument, and illustrates the use of matrix and vector notation in organizing the bookkeeping work associated with such a problem.

The analysis presented here considers the dynamic performance of the rate gyro; the rate gyro is mentioned briefly in Part 2, Section VI, as a gyroscopic instrument which may be used for measuring the angular velocity of a controlled member which is to be isolated from the motion of a rotating platform. In the following analysis, Newton's Second Law is applied first to the gyroscope rotor and then to the gimbal in which the rotor is mounted. The interaction between these two bodies produces a motion of the gimbal which can be used to measure either angular velocity or its time-integral. Both cases are considered and their salient characteristics noted. Finally, the nature and magnitude of the errors associated with these devices are discussed briefly.

In vector terms, Newton's Second Law for a rotational body may be expressed as

$$\overline{M} = D_i\overline{H} \tag{124}$$

where $\overline{M}$ is the vector sum of the torques applied to the body, $\overline{H}$ is the angular momentum of the body, and $D_i\overline{H}$ is the time rate of change of $\overline{H}$ with respect to some chosen Newtonian frame.[39] Further, $\overline{H}$ may be expressed as

$$\overline{H} = J\overline{W} \tag{125}$$

where J is a matrix array representing the moments and products of inertia of the body, and $\overline{W}$ is its angular velocity with respect to the selected Newtonian frame. J is called the inertia tensor,[40] which may be written as

$$J = \begin{vmatrix} J_{xx} & J_{xy} & J_{xz} \\ J_{yx} & J_{yy} & J_{yz} \\ J_{zx} & J_{zy} & J_{zz} \end{vmatrix}. \tag{126}$$

The terms J_{xx}, J_{yy}, and J_{zz} are the moments of inertia of the body about the x, y, and z axes, respectively, and the remaining terms are referred to as the products of inertia. If the body is considered to be a homogeneous solid, then

$$J_{xx} = \int_B (y^2 + z^2)dm$$

$$J_{yy} = \int_B (x^2 + z^2)dm$$

$$J_{zz} = \int_B (x^2 + y^2)dm$$

$$J_{xy} = \int_B xydm = J_{yx}$$

$$J_{xz} = \int_B xzdm = J_{xz}$$

$$J_{yz} = \int_B yzdm = J_{zy}$$

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[37] C. S. Draper, W. Wrigley, and L. R. Grohe, "The Floating Integrating Gyro and Its Application to Geometrical Stabilization Problems on Moving Bases," Institute of Aeronautical Sciences, New York, N. Y., Rept. No. 503; January, 1955.

[38] H. Goldstein, "Classical Mechanics," Addison-Wesley, Reading, Mass., pp. 56–176; 1950.

[39] As described in Part 2, Section VI, a Newtonian frame is a reference system in which Newton's Second Law is applicable. See J. L. Synge and B. A. Griffith, "Principles of Mechanics," McGraw-Hill Book Co., Inc., New York, N. Y., p. 32; 1949.

[40] Goldstein, *op. cit.*, p. 146 ff.

where dm is the element of mass, and the symbol $\int_B$ indicates that the integration is to be carried out over the body. [If the body is composed of n discrete particles rigidly connected, $J_{xx}=\sum_i^n (y_i^2+z_i^2)m_i$, etc.[41]] It can be shown that for any body there is a set of Cartesian coordinates such that if the inertia tensor of the body is defined with respect to these coordinates, the product of inertia terms (J_{xy}, J_{xz}, etc.) vanish. Such a set of axes is called principal axes, and it should be observed that for a symmetrical body like a gimbal frame, or a body of revolution like a gyro rotor, it is a relatively simple matter to pivot these bodies so that the pivot axis and a principal axis coincide.

From (125) it follows that (124) may be written as

$$\overline{M} = D_i(J\overline{W}). \tag{124a}$$

The central problem here is to evaluate $D_i(J\overline{W})$ in terms of measurable quantities, and equate this term to the torques applied to the body. The details of these expansions are discussed further in the following sections.

X. The Single Gimbal Gyro—The Rotor[42]

In Part 2, Section VI, the stabilization of a controlled member with a two-gimbal mount was analyzed on the basis of the "on-mount" stabilization concept; implicit in the analysis is the existence of two rate gyros attached to the controlled member, one to measure the y component of the angular velocity of the controlled member, the other the z component. Fig. 17 shows the space flow diagram associated with the controlled member and the gyro gimbals. S_a is a coordinate set attached to the controlled member, and it is presumed that the angular rate of S_a with respect to S_i, the Newtonian reference frame, may be expressed as

$$\overline{W}_{ia}{}^a = \begin{vmatrix} W_{CL} \\ W_e \\ W_t \end{vmatrix}.$$

S_e is a coordinate set attached to the gimbal of the gyro used to measure the y component (elevation) of the angular velocity of the controlled member and S_t is a coordinate set attached to the gimbal of the gyro measuring the z component (traverse). Both S_e and S_t are rotated about an axis parallel to x_a, the controlled line of the controlled member. S_e is rotated through the angle Ge and S_t is rotated through the angle Gt.

Only the elevation gyro (measuring the y component of the controlled member's angular velocity) is analyzed here. The traverse gyro differs from the elevation only in that its spin axis is directed along the y axis of S_a, whereas the spin axis of the elevation gyro is directed along the z axis of S_a. Therefore, the analysis of the traverse gyro follows the same form as that of the elevation gyro which is presented here; indeed, once the equations describing the performance of the elevation gyro have been found, similar equations for the traverse gyro may be written by inspection.

The single gimbal gyro consists of two moving parts: the gimbal, which rotates with respect to the case (the controlled member), and the gyro rotor which is supported by the gimbal. Following the usual practice of mechanical analysis, each of these elements will be considered in turn as a free body. Applying Newton's second law first to the gyro rotor, or wheel, we have

$$\overline{M}_r{}^e = D_i\overline{H}_r{}^e = D_i(J_r\overline{W}_{ir}{}^e) \tag{127}$$

where the superscript e indicates that the vectors are to be expressed in S_e, the Cartesian set attached to the elevation gyro gimbal. The operator $D_i(\ )$ indicates that we seek the time rate of change of $\overline{H}_r$ with respect to inertial space. $\overline{H}_r$ is the angular momentum of the rotor, equal to $J_r\overline{W}_{ir}$, where J_r is the inertia tensor of the rotor, and $\overline{W}_{ir}$ is the angular velocity of the rotor (S_r) with respect to the Newtonian or inertial frame (S_i). $\overline{M}_r$ represents the torque applied to the rotor by the gimbal, which may be written as

$$\overline{M}_r{}^e = \begin{vmatrix} M_{rx}{}^e \\ M_{ry}{}^e \\ M_{rz}{}^e \end{vmatrix}. \tag{128}$$

Reference to Fig. 18, which is a line diagram of the single gimbal gyro, shows that $M_{rx}{}^e$ and $M_{ry}{}^e$ are applied to the rotor by the gimbal bearings. $M_{rz}{}^e$ is the torque applied by the rotor motor, which must overcome the

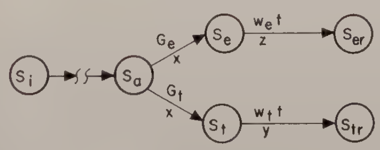

Fig. 17—Space flow diagram—two single-gimbal gyros.

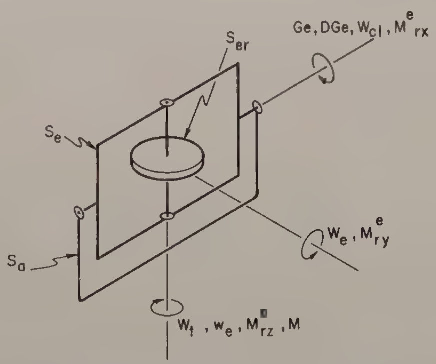

Fig. 18—Single-gimbal gyro (elevation).

[41] *Ibid.*, p. 150.

[42] Frequently called the single degree of freedom gyro by some American writers. However, since the gyro wheel turns with respect to the gimbal, albeit at a constant rate, some writers (*e.g.*, see V. A. Pavlov, "Aviation gyroscope instruments," Moscow, 1954; translated by ATIC, Wright-Patterson AFB, Ohio, available as ASTIA Document AD12226, 12227, and 12228) refer to the single gimbal gyro as a two degree of freedom gyro; to avoid confusion the term "single gimbal gyro" is used here.

friction and windage of the rotor, which need not be considered at this time.

The inertia tensor of the rotor, or gyro wheel, may be expressed as

$$J_r = \begin{vmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J_{sp} \end{vmatrix}. \tag{129}$$

Because the gyro rotor is a body of revolution, formed by the rotation about the z axis, it follows from the definitions given with (126) that $J_{xx}=J_{yy}=J$. That is, since $J_{xx}=\int_B(y^2+z^2)dm$, and since, by rotation, $x=y$, $J_{xx}=\int_B(x^2+z^2)dm=J_{yy}$. The rotor axis is a principal axis, so that the products of inertia vanish.[43] This can be seen by noting first that the body is symmetric with respect to the z axis; then, in each of the product of inertia terms, if one of the variables is considered fixed, there is both a positive and a negative value of either the x or y terms. Therefore, integrating over the body yields a net value of zero. The rotor is spinning about its z or spin axis, so the J_{zz} term is designated as J_{sp}.

The vector $\overline{W}_{ir}{}^{e}$ may be written as

$$\overline{W}_{ir}{}^{e} = \overline{W}_{ia}{}^{e} + \overline{W}_{ae}{}^{e} + \overline{W}_{er}{}^{e} \tag{130}$$

where $\overline{W}_{ia}$ is the angular velocity of S_a with respect to S_i, $\overline{W}_{ae}$ is the angular velocity of S_e with respect to S_a, and $\overline{W}_{er}$ is the angular velocity of S_{er} with respect to S_e. Or, literally, $\overline{W}_{ia}$ is the angular rate of the controlled member, designated by S_a, with respect to the Newtonian frame, designated by S_i, etc.

The three terms of (130) may each be expanded as follows:

$$\overline{W}_{ia}{}^{e} = T_{ea}\overline{W}_{ia}{}^{a} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos Ge & \sin Ge \\ 0 & -\sin Ge & \cos GE \end{vmatrix} \begin{vmatrix} W_{CL} \\ W_e \\ W_t \end{vmatrix}$$

$$= \begin{vmatrix} W_{CL} \\ W_e \cos Ge + W_t \sin Ge \\ -W_e \sin Ge + W_t \cos Ge \end{vmatrix} = \begin{vmatrix} W_1 \\ W_2 \\ W_3 \end{vmatrix} \tag{131}$$

where W_{CL}, W_e, and W_t represent the components of $\overline{W}_{ia}$ in S_a about the controlled line, the elevation axis, and the traverse axis, respectively, and W_1, W_2, and W_3 represent the x, y, and z components, respectively, of $\overline{W}_{ia}$ in S_e, the coordinates which are attached to the elevation gyro gimbal.

Also,

$$\overline{W}_{ae}{}^{e} = \begin{vmatrix} DGe \\ 0 \\ 0 \end{vmatrix}. \tag{132}$$

That is, the angular rate of the elevation gyro gimbal (S_e) with respect to the controlled member (S_a) is directed along the x_a, x_e axis, with the magnitude DGe.

Lastly,

$$\overline{W}_{er}{}^{e} = \begin{vmatrix} 0 \\ 0 \\ w \end{vmatrix} \tag{133}$$

where $w=w_e$, the constant angular rate of the gyro rotor, directed along the z_e, z_w axis.

The right-hand side of (127) can be expanded as follows by the use of the law of Coriolis (Part 2, Appendix II):

$$D_i(J_r\overline{W}_{ir}{}^{e}) = D_r(J_r\overline{W}_{ir}{}^{e}) + \overline{W}_{ir}{}^{e} \times (J_r\overline{W}_{ir}{}^{e}). \tag{134}$$

Since J_r is invariant when viewed from S_r, $D_rJ_r\equiv 0$, and the first term on the right-hand side of (134) becomes

$$D_r(J_r\overline{W}_{ir}{}^{e}) = J_r(D_r\overline{W}_{ir}{}^{e}) \tag{135}$$

and

$$D_r\overline{W}_{ir}{}^{e} = D_i\overline{W}_{ir}{}^{e} + \overline{W}_{ri}{}^{e} \times \overline{W}_{ir}{}^{e} = D_i\overline{W}_{ir}{}^{e}, \tag{136}$$

(136) can be evaluated by noting that

$$D_i(\overline{W}_{ir}{}^{e}) = D_i(T_{ea}\overline{W}_{ia}{}^{a}) + D_e\overline{W}_{ar}{}^{e} + \overline{W}_{ie}{}^{e} \times \overline{W}_{ar}{}^{e}. \tag{137}$$

Each of these expressions can be expanded as follows:

$$D_i(T_{ea}\overline{W}_{ia}{}^{a}) = D(T_{ea})\overline{W}_{ia}{}^{a} + T_{ea}D_i(\overline{W}_{ia}{}^{a})$$

$$= DGe \begin{vmatrix} 0 & 0 & 0 \\ 0 & -\sin Ge & \cos Ge \\ 0 & -\cos Ge & -\sin Ge \end{vmatrix} \begin{vmatrix} W_{CL} \\ W_e \\ W_t \end{vmatrix}$$

$$+ \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos Ge & \sin Ge \\ 0 & -\sin Ge & \cos GE \end{vmatrix} \begin{vmatrix} A_{CL} \\ A_e \\ A_t \end{vmatrix}$$

$$= \begin{vmatrix} A_1 \\ A_2 + DGe(W_3) \\ A_3 - DGe(W_2) \end{vmatrix} \tag{138}$$

where, following the notation used in (131), A_{CL}, A_e, and A_t are the components of the angular acceleration of the controlled member in S_a, and A_1, A_2, and A_3 are the components of angular acceleration of the controlled member in S_e. That is, $A_1=A_{CL}$, $A_2=A_e \cos Ge+A_t \sin Ge$, and $A_3=-A_e \sin Ge+A_t \cos Ge$.[44]

Also, the next term in (137) may be written as

$$D_e\overline{W}_{ar}{}^{e} = D\begin{vmatrix} DGe \\ 0 \\ w \end{vmatrix} = \begin{vmatrix} D^2Ge \\ 0 \\ 0 \end{vmatrix}. \tag{139}$$

[43] Goldstein, *op. cit.*, pp. 149–156.

[44] See (121)–(123), Part 2, Appendix II, for typical forms for A_{CL}, A_e, and A_t.

(Note that $Dw=0$ because the synchronous rotor motor acts to keep w=constant.) Lastly, the third term in (137) may be expanded as follows:

$$\overline{W}_{ie}{}^{e} \times \overline{W}_{ar}{}^{e} = \begin{vmatrix} W_1 + DGe \\ W_2 \\ W_3 \end{vmatrix} \times \begin{vmatrix} DGe \\ 0 \\ w \end{vmatrix}$$

$$= \begin{vmatrix} w(W_2) \\ DGe(W_3) - w(W_1 + DGe) \\ -DGe(W_2) \end{vmatrix}. \tag{140}$$

Eqs. (138)–(140) may be substituted into (137) to give

$$D_i(\overline{W}_{ir}{}^{e}) = \overline{A}_{ir}{}^{e}$$

$$= \begin{vmatrix} A_1 + D^2Ge + wW_2 \\ A_2 + 2DGe(W_3) - w(W_1 + DGe) \\ A_3 - 2DGe(W_2) \end{vmatrix} \tag{137a}$$

where $\overline{A}_{ir}{}^{e}$ is the angular acceleration of the gyro rotor with respect to inertial space, expressed in gimbal coordinates. Multiplying (137a) by J_r gives the first term on the right-hand side of (134), as follows:

$$\begin{vmatrix} J[A_1 + D^2Ge + w(W_2)] \\ J[A_2 + 2DGe(W_3) - w(W_1 + DGe)] \\ J_{sp}[A_3 - 2DGe(W_2)] \end{vmatrix}. \tag{141}$$

The second term on the right-hand side of (134) may be written

$$\overline{W}_{ir}{}^{e} \times (J_w \overline{W}_{ir}{}^{e})$$

$$= \begin{vmatrix} W_1 + DGe \\ W_2 \\ W_3 + w \end{vmatrix} \times \begin{vmatrix} J & 0 & 0 \\ 0 & J & 0 \\ 0 & 0 & J_{sp} \end{vmatrix} \begin{vmatrix} W_1 + DGe \\ W_2 \\ W_3 + w \end{vmatrix}$$

$$= \begin{vmatrix} (J_{sp} - J)W_2(W_3 + w) \\ (J - J_{sp})(W_1 + DGe)(W_3 + w) \\ (J - J)(W_1 + DGe)W_2 \end{vmatrix}. \tag{142}$$

Therefore, (141) and (142) may be combined to evaluate (134), which may be equated to (128), giving

where $H_{sp} = J_{sp}w$.

Eq. (128a) relates the moments applied to the gyro wheel to its angular motion. Before proceeding to a description of the motion of the gimbal, it is of interest to consider a simplified form of (128a). Note that

$$\overline{H}_r{}^{e} = J_r \overline{W}_{ir}{}^{e} = \begin{vmatrix} J(W_1 + DGe) \\ JW_2 \\ J_{sp}(W_3 + w) \end{vmatrix}$$

$$= H_{sp} \begin{vmatrix} \dfrac{J}{H_{sp}}(W_1 + DGe) \\ \dfrac{J}{H_{sp}} W_2 \\ \dfrac{W_3}{w} + 1 \end{vmatrix}. \tag{143}$$

In a practical gyroscope, w is a large number, so that $W_3/w \ll 1$, and $J/H_{sp} \to 0$. Then (143) may be written

$$\overline{H}_r{}^{e} \approx H_{sp} \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}. \tag{143a}$$

Eq. (128) may be expanded as follows:

$$\overline{M}_r{}^{e} = D_i \overline{H}_r{}^{e} = D_r \overline{H}_r{}^{e} + \overline{W}_{ir}{}^{e} \times \overline{H}_r{}^{e}$$

$$= 0 + \begin{vmatrix} W_1 + DGe \\ W_2 \\ W_3 + w \end{vmatrix} \times H_{sp} \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}$$

$$= H_{sp} \begin{vmatrix} W_2 \\ -(W_1 + DGe) \\ 0 \end{vmatrix}. \tag{128b}$$

Only the x component of (128a) or (128b) is of interest to us in this discussion. It follows from the diagram of the gyro (Fig. 18) that the x axis of the gimbal is the "output" axis; that is, the motion of the gimbal about that axis is measurable, and, as is shown below, this measurement is useful as a control signal in the geometric stabilization systems described in Part 2. $M_{rx}{}^{e}$ is the torque which the gimbal applies to the rotor; therefore, the reaction torque that the rotor applies to

$$\begin{vmatrix} M_{rx}{}^{e} \\ M_{ry}{}^{e} \\ M_{rz}{}^{e} \end{vmatrix} = D_i(J_r \overline{W}_{ir}{}^{e}) = \begin{vmatrix} J(A_1 + D^2Ge) + (J_{sp} - J)W_2W_3 + H_{sp}W_2 \\ J(A_2 - 2DGeW_3) + (J - J_{sp})W_3(W_1 + DGe) + H_{sp}(W_1DGe) \\ J_{sp}(A_3 - 2DG3W_2) \end{vmatrix} \tag{128a}$$

the gimbal, M_r, is equal and opposite to M_{rx}^e, and it follows that

$$M_r = -M_{rx}^e = -J(A_1 + D^2Ge) - (J_{sp} - J)W_2W_3 - H_{sp}W_2$$
$$= -H_{sp}\left[J/H_{sp}(A_1 + D^2Ge) + W_2\left(1 + W_3\left(\frac{J_{sp} - J}{H_{sp}}\right)\right)\right] \quad (144a)$$
$$\approx -H_{sp}W_2 = -H_{sp}(W_e \cos Ge + W_t \sin Ge) \quad (144b)$$

since $J/H_{sp} \to 0$ and $W_3/w \ll 1$.

The next section considers how this torque and other torques applied to the gyro gimbal affect the motion of the gimbal.

XI. The Single Gimbal Gyro—Gimbal Motion

Applying Newton's second law to the elevation gyro gimbal, we have

$$\overline{M}_e^e = D_i\overline{H}_e^e = D_i(J_e\overline{W}_{ie}^e) \quad (145)$$

where, as before, the superscript e indicates that the vectors are to be evaluated in terms of their components in S_e, which is the coordinate set attached to the elevation gyro gimbal. $\overline{M}_e$ is the vector sum of all the moments applied to the elevation gyro gimbal, J_e is the inertia tensor of the elevation gyro gimbal, and $\overline{W}_{ie}$ is the angular rate of the gimbal (S_e) with respect to the inertial frame (S_i).

Consider first the moment vector. $\overline{M}_e^e$ may be expanded as

$$\overline{M}_e^e = \begin{vmatrix} M_c + M_r - CDGe - KGe \\ M_{ey}^e \\ M_{ez}^e \end{vmatrix}. \quad (146)$$

M_c is the control torque applied to the gimbal. M_r is the gyroscopic torque applied to the gimbal by the spinning gyro rotor, which has been developed in the previous section. C is a damping constant and $CDGe$ is the damping torque generated by relative motion between the gyro gimbal and its case, which is fixed to the controlled member. K is a spring constant, and KGe is a restraining torque generated by the deflection of the gimbal with respect to the case. M_{ey}^e and M_{ez}^e are the restraining torques developed by the gimbal support bearings.

The inertia tensor for the gimbal, which appears in (145), may be written as

$$J_e = \begin{vmatrix} J_x & 0 & 0 \\ 0 & J_y & 0 \\ 0 & 0 & J_z \end{vmatrix} \quad (147)$$

in which, by arguments similar to those used above in the discussion of the gyro rotor, the products of inertia vanish because of the symmetry of the gimbal.

Proceeding as before, the right-hand side of (145) may be expanded as follows:

$$D_i(J_e\overline{W}_{ie}^e) = D_e(J_e\overline{W}_{ie}^e) + \overline{W}_{ie}^e \times (J_e\overline{W}_{ie}^e). \quad (148)$$

$D_i(J_e\overline{W}_{ie}^e)$ is written in this form so that the vector can be expanded in terms of its components, all of which represent measurable quantities. Eq. (148) represents an application of the law of Coriolis, and it may be recalled that $D_eJ_e \equiv 0$. Therefore, the first term on the right-hand side of (148) becomes

$$D_e(J_e\overline{W}_{ie}^e) = J_eD_e(\overline{W}_{ie}^e). \quad (149)$$

But we may observe that

$$D_e(\overline{W}_{ie}^e) = D_i(\overline{W}_{ie}^e) + \overline{W}_{ie}^e \times \overline{W}_{ie}^e$$
$$= D_i(\overline{W}_{ie}^e) = \overline{A}_{ie}^e \quad (150)$$

where $\overline{A}_{ie}$ is the angular acceleration of S_e with respect to S_i. Eq. (150) can be evaluated in terms of its S_e components as shown in Section X (see also Part 2, Appendix II) to give

$$\overline{A}_{ie}^e = \begin{vmatrix} A_1 + D^2GE \\ A_2 + 2DGeW_3 \\ A_3 - 2DGeW_2 \end{vmatrix} \quad (150a)$$

so that (149) becomes

$$J_e\overline{A}_{ie}^e = \begin{vmatrix} J_x(A_1 + D^2Ge) \\ J_y(A_2 + 2DGeW_3) \\ J_z(A_3 - 2DGeW_2) \end{vmatrix}. \quad (151)$$

The second term of (148) is given by

$$\begin{vmatrix} W_1 + DGe \\ W_2 \\ W_3 \end{vmatrix} \times \begin{vmatrix} J_x(W_1 + DGe) \\ J_y(W_2) \\ J_z(W_3) \end{vmatrix} = \begin{vmatrix} (J_z - J_y)(W_2W_3) \\ (J_x - J_z)(W_3)(W_1 + DGe) \\ (J_y - J_x)(W_2)(W_1 + DGe) \end{vmatrix} \quad (152)$$

so that (145) becomes

$$\overline{M}_e^e = \begin{vmatrix} J_x(A_1 + D^2Ge) + (J_z - J_y)W_2W_3 \\ J_y + (2DGeW_2) + (J_x - J_z)W_3(W_1 + DGe) \\ J_z(A_3 - 2DGeW_3) + (J_y - J_x)W_2(W_1 + DGe) \end{vmatrix}. \quad (145a)$$

Only the x component of (145a)[45] is of interest here because the output of the gyro is the rotation Ge about x_e; therefore,

[45] It should be observed that the torques M_{yb} and M_{zb} are couples produced by the gimbal bearings. The radial forces on the bearings associated with these couples also produce frictional torques about x_e. These effects are not considered here; for further information on the magnitudes of these torques and other aberrations, see H. E. Soland, "Floated Gyro Application Notes No. 4," Minneapolis-Honeywell Regulator Co., Minneapolis, Minn., "Aero Document U-ED9749; 1955.

$$M_c + M_r - CDGe - KGe = J_x(A_1 + D^2Ge) + (J_z - J_y)W_2W_3. \quad (153)$$

Eq. (153) is the basic differential equation which describes the operation of the rate gyro. From this equation the operation of the integrating rate gyro can also be deduced, since the integrating rate gyro is a special case of the rate gyro for which

$$C \gg J_x \qquad K \to 0.$$

These relations are considered in more detail in the following section, along with a discussion of the errors associated with the operation of the rate and integrating rate gyros.

XII. The Rate Gyro

Consider first the rearrangement of (153) where the gyroscopic rotor torque as given by (144a) has been substituted for M_r:

$$\begin{aligned}(J_x + J) D^2Ge + CDGe + KGe &= M_c - H_{sp}W_2 - (J_x + J) A_1 \\ &\quad - [(J_z + J_{sp}) - (J_y + J)]W_2W_3. \qquad (154a)\end{aligned}$$

Using (144b) instead of (144a), (153) reduces to

$$\begin{aligned}J_xD^2Ge + CDGe + KGe = M_c - H_{sp}W_2 + J_xA_1 \\ - (J_z - J_y)W_2W_3. \qquad (154b)\end{aligned}$$

Eqs. (154a) and (154b) have the same form, but it is clear, by comparing their left-hand sides, that the usual gyro torque approximation, as represented by (144b), results in some error in the evaluation of the dynamic characteristics of the gyro. That is, (154b) considers only the inertia of the gimbal, and neglects the inertia of the gyro rotor. Therefore, we shall continue, in this discussion, to use the more exact expression given by (154a).

Substituting for A_1, W_2, and W_3, (154a) becomes

$$\begin{aligned}\bar{J}_xD^2Ge + CDGe + KGe \\ = M_c - H_{sp}W_e + E_1 + E_2 + E_3 + E_4 \quad (155)\end{aligned}$$

where E_1, E_2, E_3, and E_4 are error quantities, defined as follows:

$$\begin{aligned}E_1 &= H_{sp}W_e(1 - \cos Ge) \\ E_2 &= - H_{sp}W_e \sin Ge \\ E_3 &= \bar{J}_xA_{CL} \\ E_4 &= (\bar{J}_z - \bar{J}_y)(-W_e \sin Ge + Wt \cos Ge) \\ &\quad \cdot (W_e \cos Ge + W_t \sin Ge)\end{aligned}$$

and

$$\begin{aligned}\bar{J}_x &= J_x + J \\ \bar{J}_y &= J_y + J \\ \bar{J}_z &= J_z + J_{sp}.\end{aligned}$$

In a base motion isolation system, the output of the gyro is the gimbal deflection angle Ge, suitably measured by an electrical pick-off device. This electrical signal is applied to a servo which controls W_e in such a way that $W_e \approx M_r/H_{sp}$. Therefore, neglecting errors, Ge approaches zero in the steady state. Therefore, E_1 and E_2 are very small and approach zero in the steady state.

Two effects tend to minimize E_3: in the first place, $\bar{J}_x/H_{sp} \ll W_e$, and in the second place, for a three-gimbal mount, $A_{CL} \to 0$ in the steady state.

As has been shown, Ge goes to zero in the steady state, so that $E_4 \approx (\bar{J}_z - \bar{J}_y)\ W_eW_t$; for H_{sp} large, as occurs in a practical gyroscope, $(\bar{J}_z - \bar{J}_y) W_t/H_{sp} \ll 1$, so that E_4, also, can be neglected.

Consider next that $K \equiv 0$. Then, neglecting the error terms, (155) becomes

$$Ge = \frac{K_i}{D(TD + 1)}(W_c - W_e) \qquad (156)$$

where

K_i = integrating gyro sensitivity $= H_{sp}/C$,
T = integrating gyro time constant $= \bar{J}_x/C$,
W_c = command rate $= M_c/H_{sp}$.

The definition $W_c = M_c/H_{sp}$ illustrates the relationships between the torque applied to the gyro, its angular rate, and the angular momentum of the gyro. Also, as can be seen from (156), this definition shows how the torque-summing nature of the gyro gimbal is used to compare the controlled angular velocity (in this case, W_e), with the commanded angular velocity, W_c.

For a conventional rate gyro, $K \neq 0$, and neglecting errors, (156) becomes

$$Ge = \frac{K_r}{\dfrac{D^2}{W_n} + 2\xi\dfrac{D}{W_n} + 1}(W_c - W_e) \qquad (157)$$

where

$$K_r = \text{rate gyro sensitivity}^{46} = H_{sp}/K,$$

$$W_n = \text{rate gyro undamped natural frequency} = \sqrt{K/\bar{J}_x},$$

$$\xi = \text{rate gyro damping coefficient} = \frac{C}{2\sqrt{J_xK}},$$

$$W_c = \frac{M_c}{H_{sp}}.$$

Several observations can be made concerning (156) and (157). First of all, the transfer function for a con-

[46] In a practical application, the volts/radian sensitivity of the pick-off device must be included with K_i and K_r.

ventional rate gyro consists of a constant over a second-order lag. The transfer function for an integrating gyro consists of a constant over the product of a first-order lag and an integration stage. It is this integration stage which is of such great importance in the design of stabilization systems, because of its effect on the reduction of forced errors in the stabilization loop.

Further, it may be noted that the output of both the rate and integrating rate gyros is proportional to the difference between the command quantity W_c and the output or measured quantity W_e. Thus the gyro may be used as an error detection device; this property is not commonly used on ordinary rate gyros, but it is very useful in instrumenting the computations required in fire control and inertial navigation systems.

XIII. The Two-Gimbal Gyro[47]

The characteristics of two-gimbal gyros have been described extensively elsewhere;[48] however, a brief description of these devices is given here to demonstrate certain characteristics of those gyros which make them so useful for stabilization systems.

Fig. 19 shows a space flow diagram which describes the gimbal arrangements associated with the vertical gyro and the directional gyro. Fig. 20 is a line diagram illustrating the gimbal arrangement of the vertical gyro, and Fig. 21 is a line diagram showing the gimbal arrangement of the directional gyro. Note that the spin axis of the vertical gyro rotor is vertical, and that of the directional gyro rotor is horizontal. We shall examine the behavior of each of these two gyros in the following manner. Let the correct or ideal orientation of the gyro rotor spin axis represent a "line of sight" (LS). Let the actual orientation represent a "tracking line" (TL). Next, consider the rotation of the gimbal axes required to align the tracking line with the line of sight for various platform Euler angles; finally, we shall determine the significance of these gimbal deflections as signals useful for controlling the attitude of some platform in inertial space.

Referring now to Fig. 19, S_i represents the inertial reference frame. H, R, and P are the three Euler angles which specify the attitude of the platform, S_p, with respect to S_i. The symbols S_h and S_r represent auxiliary sets of coordinates. The gyro spin axes are referred to S_d by the symbols S_{xy}, G_{xy}, where the S's represent coordinate frames attached to the two gimbals and the G's represent the angles through which the gimbals ro-

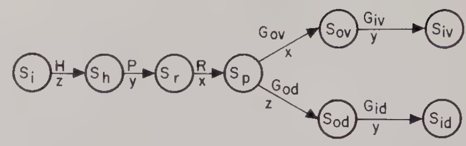

Fig. 19—Space flow diagram—vertical and directional gyros.

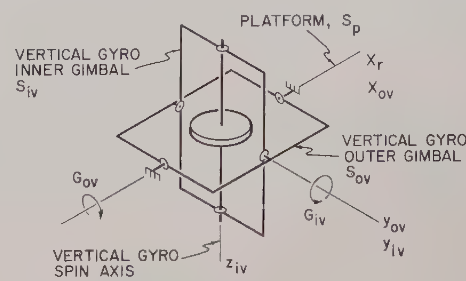

Fig. 20—Vertical gyro.

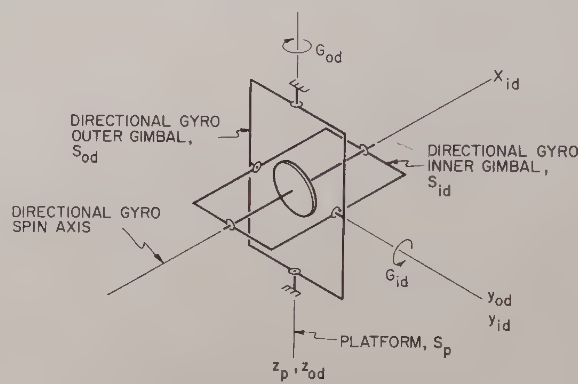

Fig. 21—Directional gyro.

tate. The subscripts associated with S and G are defined as follows:

$$x = \begin{cases} o \text{ for outer gimbal} \\ i \text{ for inner gimbal} \end{cases}$$

$$y = \begin{cases} v \text{ for vertical gyro} \\ d \text{ for directional gyro.} \end{cases}$$

Consider first the line of sight vector associated with the vertical gyro. Let the vector be represented by the symbol $\overline{LS_v}$, and in S_i the components of the vector are given by

$$\overline{LS_v}^i = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}.$$

Also, let the tracking line be represented by $\overline{TL_v}$, and in S_{iv}

$$\overline{TL_v}^{iv} = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}. \tag{158}$$

[47] Following a note by R. C. Webber.

[48] L. Becker, "Gyro pickoff indications at arbitrary plane attitudes," *J. Aeronaut. Sciences*, vol. 18, pp. 718–724; November, 1951.
M. S. Abzug, "Application of matrix operators for the kinematics of airplane motion," *J. Aeronaut. Sciences*, vol. 23, pp. 679–684; July, 1956, and Pavlov, *op. cit.*, footnote 42, chs. 1, 3, 4.

Now, we desire that

$$\overline{TL}_v = \overline{LS}_v \tag{158}$$

or

$$\overline{TL}_v^{iv} = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = T_{iv-i} \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix} = \overline{LS}_v^{iv} \tag{158a}$$

where T_{iv-i} is the transformation matrix which transforms $\overline{LS}_v$ from S_i to S_{iv}. From Part 2, Appendix I, we know that T_{iv-i} has the following characteristics:

$$T_{iv-v} = \begin{vmatrix} \alpha_1 & \alpha_2 & 0 \\ \beta_1 & \beta_2 & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

from which we may determine the relationships between G_{ov}, G_{iv} and H, P, and R so that behavior of the vertical gyro can be determined.

We note that

$$T_{hi} = \begin{vmatrix} \cos H & \sin H & 0 \\ -\sin H & \cos H & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

so that

$$\overline{LS}_v^h = T_{hi}\overline{LS}_v^i = \begin{vmatrix} 0 \\ 0 \\ 1 \end{vmatrix}. \tag{159}$$

Also,

$$T_{rh} = \begin{vmatrix} \cos P & 0 & -\sin P \\ 0 & 1 & 0 \\ \sin P & 0 & \cos P \end{vmatrix}$$

so that

$$\overline{LS}_v^r = T_{rh}\overline{LS}_v^h = \begin{vmatrix} -\sin P \\ 0 \\ \cos P \end{vmatrix}$$

and

$$T_{pr} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos R & \sin R \\ 0 & -\sin R & \cos R \end{vmatrix}$$

giving

$$\overline{LS}_v^p = T_{pr}\overline{LS}_v^r = \begin{vmatrix} -\sin P \\ \sin R \cos P \\ \cos R \cos P \end{vmatrix} = \begin{vmatrix} x_p \\ y_p \\ z_p \end{vmatrix}. \tag{160}$$

That is, the unit vector $\overline{LS}$, when expressed in its S_p or platform coordinates, has the following components:

$$x_p = -\sin P$$
$$y_p = \sin R \cos P$$
$$z_p = \cos R \cos P.$$

It is interesting to observe that at every transformation step, the magnitude of the vector remains unchanged, as can be seen by writing

$$\begin{aligned} |\overline{LS}| &= x_p^2 + y_p^2 + z_p^2 \\ &= \sin^2 P + \cos^2 P(\sin^2 R + \cos^2 R) \\ &= 1. \end{aligned}$$

Now, since

$$T_{ov-p} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \cos G_{ov} & \sin G_{ov} \\ 0 & -\sin G_{ov} & \cos G_{ov} \end{vmatrix},$$

it follows that

$$\overline{LS}_v^{ov} = T_{ov-p} \begin{vmatrix} x_p \\ y_p \\ z_p \end{vmatrix}$$
$$= \begin{vmatrix} x_p \\ y_p \cos G_{ov} + z_p \sin G_{ov} \\ -y_p \sin G_{ov} + z_p \cos G_{ov} \end{vmatrix} = \begin{vmatrix} x_{ov} \\ y_{ov} \\ z_{ov} \end{vmatrix}. \tag{161}$$

Since the rotation G_{iv} occurs about the common y_{ov}, y_{iv} axis, the y_{iv} component of $\overline{LS}_v$ must be equal to the y_{ov} component. But if $\overline{LS}_v = \overline{TL}_v$, y_{iv} must be zero [see (158a)]. Therefore,

$$G_{ov} = \tan^{-1}(-y_p/z_p). \tag{162}$$

Substituting from (160) for y_p and z_p gives

$$G_{ov} = \tan^{-1}(-\tan R)$$

or

$$G_{ov} = -R. \tag{162a}$$

Therefore, (161) becomes

$$\overline{LS}_v^{ov} = \begin{vmatrix} -\sin P \\ 0 \\ \cos P \end{vmatrix} \tag{161a}$$

and, since

$$T_{iv-ov} = \begin{vmatrix} \cos G_{iv} & 0 & -\sin G_{iv} \\ 0 & 1 & 0 \\ \sin G_{iv} & 0 & \cos G_{iv} \end{vmatrix},$$

it follows that

$$\overline{LS}_v^{iv} = T_{iv-ov}\overline{LS}_v^{ov}$$
$$= \begin{vmatrix} -\sin P \cos G_{iv} - \cos P \sin G_{iv} \\ 0 \\ -\sin P \sin G_{iv} + \cos P \cos G_{iv} \end{vmatrix}. \quad (163)$$

Therefore, if (158a) is to be satisfied, it follows that

$$G_{iv} = - P; \quad (164)$$

that is, the inner and outer gimbal angles of the vertical gyro are equal and opposite to the pitch and roll angles of the platform, respectively. This result may be anticipated by reference to the space-flow diagram (Fig. 19) wherein it can be seen that the vertical gyro gimbal angles G_{ov} and G_{iv} "unwind" the platform rotation angles R and P, since G_{ov}, like R, is about a common x axis, and G_{iv}, like P, is about a common y axis.

The procedure in the case of the directional gyro is much the same, except that the line of sight vector is given by

$$\overline{LS}_d^{i} = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$$

and we desire that

$$\overline{LS}_d^{id} = T_{id-i}\overline{LS}_d^{i}$$
$$= T_{id-i}\begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} = \overline{TL}_d^{id} = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix} \quad (165)$$

or

$$T_{id-i} = \begin{vmatrix} 1 & 0 & 0 \\ 0 & \beta_2 & \beta_3 \\ 0 & \gamma_2 & \gamma_3 \end{vmatrix}. \quad (165a)$$

Proceeding as before, we note that

$$\overline{LS}_d^{p} = \begin{vmatrix} \cos H \cos P \\ -\sin H \cos R + \sin R \cos H \sin P \\ \sin H \sin R + \cos H \sin P \cos R \end{vmatrix} = \begin{vmatrix} x_p \\ y_p \\ z_p \end{vmatrix}. \quad (166)$$

Further,

$$T_{od-p} = \begin{vmatrix} \cos G_{od} & \sin G_{od} & 0 \\ -\sin G_{od} & \cos G_{od} & 0 \\ 0 & 0 & 1 \end{vmatrix}$$

$$T_{id-od} = \begin{vmatrix} \cos G_{id} & 0 & -\sin G_{id} \\ 0 & 1 & 0 \\ \sin G_{id} & 0 & \cos G_{id} \end{vmatrix}$$

so that

$$\overline{LS}_d^{od} = \begin{vmatrix} x_p \cos G_{od} + y_p \sin G_{od} \\ -x_p \sin G_{od} + y_p \cos G_{od} \\ z_p \end{vmatrix} = \begin{vmatrix} x_{od} \\ y_{od} \\ z_{od} \end{vmatrix}. \quad (167)$$

Since $y_{od}=y_{id}$, and y_{id} must equal zero, it follows from (165) that

$$G_{od} = \tan^{-1}(y_p/x_p). \quad (168)$$

Substituting from (166), we have

$$G_{od} = \tan^{-1}\frac{\sin R \cos H \sin P - \sin H \cos R}{\cos H \cos P}. \quad (168a)$$

It can be seen from (168a) that for $R=P=0$,

$$G_{od} = - H; \quad (169)$$

that is, if the platform, as designated by S_p, is not pitched or rolled, then the directional gyro indicates the heading H of the platform. When P and R are not zero, there is an error involved in the measurement of H. Let this error be defined as $(H-G_{od})$; then Fig. 22 shows the range of this heading error for P and R in the regions

$$0 < P < \pi/4$$
$$0 \leq R < \pi/9.$$

In some situations it is necessary to measure H precisely, independent of the values of P and R. The discussion on coordinate converters presented in Part 1 suggests how this may be done. In qualitative terms, the design procedure for such a coordinate converter may be stated as follows:

1) From (168a) and (169), we see that for $R=P=0$, $G_{od}=-H$. Therefore, we may consider G_{od} as the input to our coordinate converter, and H as the output, with R and P as disturbances.
2) Given the unit vector $\overline{LS}_d$, we see from the space flow diagram (Fig. 19) and T_{id-od} that if $y_{od}=0$, then the proper relation between H and G_{od} is satisfied independent of R and P.[49]
3) From 1) and 2) we see that the unit vector $\overline{LS}_d$ successively transformed through H, P, R, and G_{od} has a zero magnitude y_{od} component. Therefore, if P and R, obtained from a vertical gyro, and G_{od}, obtained from a directional gyro, are used to drive resolvers, H may be generated by using y_{od} as the error voltage driving a servo which positions an H resolver; when the servo is in equilibrium, *i.e.*, when $y_{od}\equiv 0$, (168a) is satisfied. Fig. 23 shows the block diagram for this system.

[49] Except, as can be seen from (168a), if $P\rightarrow\pi/2$, in which case $G_{od}\rightarrow\pi/2$, independent of H, or if

$$H = \tan^{-1}(\tan R \sin P)$$

in which case $G_{od}\rightarrow 0$, independent of H. We will disregard these trigonometric singularities since they are not necessary for the present discussion.

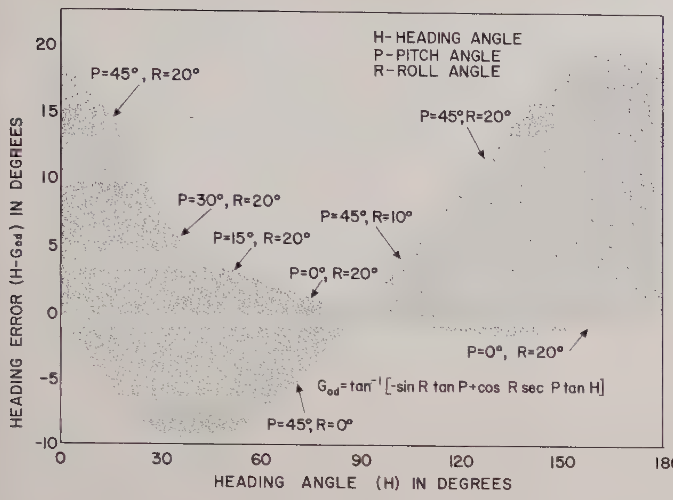

Fig. 22—Heading error ($H-G_{od}$) as a function of H, P, and R.

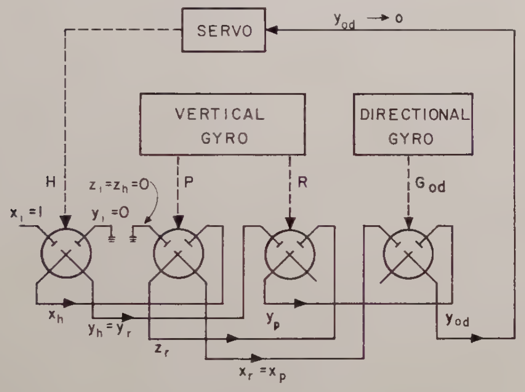

Fig. 23—Heading angle computer.

Examination of the trade literature and technical publications on gyroscopes[50] reveals many interesting gyro configurations. All of them can be analyzed with the same tools that have been used here, for fun and profit. The use of a concise vector algebra permits not only the detailed examination of gyro kinetics, as described in the case of the single gimbal gyro in Sections X–XII, but also the more elementary examination of the functions of the two gimbal gyros as presented in the present section. Of special current interest is the operation of gyro-stabilized platforms as the basic mechanism in missile guidance systems. It should be observed that the three gimbal platforms described in Part 2, Section VII, is just such a device. This three-axis platform is presented as an extension of the two-axis platform used for stabilizing a line in space. Therefore there is a direct analogy between the two-axis platform described in Part 2 and the two gimbal gyros of the present section, since both maintain the orientation of line in space. As described in Part 2, however, the three-axis platform maintains the orientation of a plane in space. Thus, it can be seen that by mounting three single gimbal gyros on a three-axis platform, we achieve a three gimbal, synthetic "gyro" capable of measuring the attitude of a body in space. Note, also, that this synthetic gyro performs the same function of the vertical and directional gyros described in this section when the two latter devices are combined with a coordinate converter as shown in Fig. 23.

Finally, in the design of such a platform, the kinetic behavior of the device is of considerable interest. Although the detailed dynamic performance of such a device is beyond the scope of the present paper, the vector algebra used in Sections X–XII can be extended to the additional supporting gimbals of the platform by successive application of the free body principle. Since the number of supporting gimbals may vary between one and four or more (corresponding, for example, to a two-gimbal gyro, or a four-gimbal stable platform), it is clear that the need for an orderly approach to the problem is considerable. In such analyses, note how the space flow diagram itself serves to organize the study and establish the necessary depth of symbology. Then, given the space flow diagram, it is relatively easy, although tedious, to write the vector equations of the form given by (124) and expand these equations in terms of the measurable physical quantities which describe the dynamic behavior of the control system.

XIV. The Accelerometer[51]

For the final application of our vector algebra, consider the equations which describe the dynamic behavior of an accelerometer. Like the gyroscope, the dynamic behavior of the accelerometer is also described by Newton's Second Law. However, in this case we must deal with Newton's Second Law for a translational system rather than for a rotational system.

The previous examples have dealt with problems involving rotation, and the space flow diagram has been used as a simple device for organizing the steps associated with the solution of these problems.

In Part 1, the space flow diagram was presented as a formal device for depicting a three-dimensional diagram. Symbols were introduced which described sets of coordinates, angular rotations, and the axes about which these rotations occured. As a corollary of this formal usage, it was demonstrated that the space flow diagram suggested the order and functions of resolvers in coordinate converters. In addition, the space flow diagram was used to establish notational requirements and to organize the operational steps necessary to solve several types of vector equations. It is the latter characteristics which we wish to extend in the present section to problems in translation motion, as opposed to the rotary motion considered in the previous sections. For example, consider the space flow diagram and vector diagram shown in Figs. 24 and 25, respectively. The

[50] See especially Pavlov, *op. cit.*, footnote 42, ch. 8.

[51] After W. Wrigley, R. B. Woodbury, and J. Hovorka, "Inertial Guidance," Institute of Aeronautical Sciences, New York, N. Y., paper FF-16; 1957.

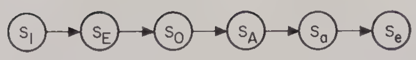

Fig. 24—Space flow diagram—accelerometer.

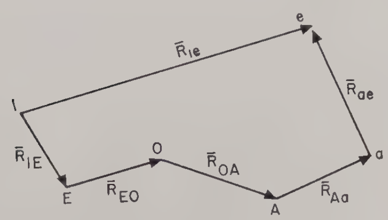

Fig. 25—Vector diagram—accelerometer.

mass element of the accelerometer is located in inertial space by the vector $\overline{R}_{Ie}$ shown in Fig. 25. Also, it is located with respect to the accelerometer case by the vector $\overline{R}_{ae}$, which in turn is located with respect to the center of mass of an aircraft, for example, by the vector $\overline{R}_{Ae}$ etc., as shown in Fig. 25.

The positions in the vector diagram labelled I, E, etc., in Fig. 25 represent a series of Cartesian coordinates designated S_1, S_E, etc. These Cartesian sets may then be defined as follows:

S_I = an arbitrary Newtonian frame,
S_E = a frame located at the center of the earth,
S_O = a frame located at a point moving in orbit about the earth,
S_A = a frame fixed to the vehicle center of mass,
S_a = a frame fixed to the accelerometer,
S_e = a frame fixed to the mass element of the acclerometer.

Then, from Fig. 24, it can be seen that the concept of the space flow diagram can be used to describe translational systems as well as rotational systems if we drop the formalism introduced in Part 1 concerning angles and axes.[52]

In the vector terms, Newton's Second Law for a translational system is given by

$$m_a D_I^2 \overline{R}_{Ia} = \sum \overline{F}_a + m_a \sum \overline{G}_a \qquad (170)$$

where

m_a = mass of object of a,
$\overline{R}_{Ia}$ = position of object a with respect to inertial space, S_I,
$D_I^2 R_{Ia}$ = acceleration of object a with respect to inertial space,
$\sum \overline{F}_a$ = sum of nonfield forces acting on object a,
$\sum \overline{G}_a$ = sum of field forces acting on object a.

[52] The formalism can in fact be extended to include both rotational and translational systems. However, this extension is not required for the present discussion.

Then, for a typical accelerometer system,

$$m_e D_I^2 \overline{R}_{Ie} = -k\overline{R}_{ae} - cD_a\overline{R}_{ae} + m_e(\overline{G}_{Ee} + \overline{G}_{Me} + \overline{G}_{Se} + \cdots) \qquad (171)$$

where

m_e = mass of accelerometer element,
k = spring constant,
$\overline{R}_{ae}$ = deflection of accelerometer mass with respect to the case,
c = damping constant,
$D_a\overline{R}_{ae}$ = velocity of mass element with respect to accelerometer case,
$\overline{G}_{Ee}, \overline{G}_{Me}, \overline{G}_{Se}$ = gravitational fields at mass element due to earth, moon, and sun, respectively.

From Figs. 24 and 25, we note that

$$\overline{R}_{Ie} = \overline{R}_{Ia} + \overline{R}_{ae} \qquad (172)$$

so that, applying the law of Coriolis, we get

$$D_I\overline{R}_{Ie} = D_I\overline{R}_{Ia} + D_a\overline{R}_{ae} + \overline{W}_{Ia} \times \overline{R}_{ae}. \qquad (173)$$

Differentiating again gives

$$D_I^2\overline{R}_{Ie} = D_I^2\overline{R}_{Ia} + D_a^2\overline{R}_{ae} + 2\overline{W}_{Ia} \times D_a\overline{R}_{ae} + (D_I\overline{W}_{Ia}) \times \overline{R}_{ae} + \overline{W}_{Ia} \times (\overline{W}_{Ia} \times \overline{R}_{ae}). \qquad (174)$$

In addition,

$$R_{Ia} = R_{IA} + \overline{R}_{Aa} \qquad (175)$$

which can be expanded in the same fashion to give

$$D_I\overline{R}_{Ia} = D_I\overline{R}_{IA} + D_A\overline{R}_{Aa} + \overline{W}_{IA} \times \overline{R}_{Aa}. \qquad (176)$$

The vector $\overline{R}_{Aa}$ locates the accelerometer case with respect to the center of mass of the airframe. Consesequently its time-rate of change as viewed from airframe coordinates is zero; that is,

$$D_A\overline{R}_{Aa} \equiv 0.$$

Therefore, the acceleration of the accelerometer case with respect to inertial space is given by

$$D_I^2\overline{R}_{Ia} = D_I^2\overline{R}_{IA} + (D_I\overline{W}_{IA}) \times \overline{R}_{Aa} + \overline{W}_{IA} \times (\overline{W}_{IA} \times \overline{R}_{Aa}). \qquad (177)$$

But from (170),

$$D_I^2\overline{R}_{IA} = \frac{\sum \overline{F}_A}{m_A} + \overline{G}_{EA} + \overline{G}_{MA} + \overline{G}_{SE} + \cdots \qquad (178)$$

where

m_A = airframe mass,
$\sum \overline{F}_A$ = sum of nonfield forces on airframe,
$\overline{G}_{EA}, \overline{G}_{MA}, \overline{G}_{SA}$ = gravitational fields of earth, moon, and sun at airframe.

The substitution of (174), (177) and (178) into (171) gives

$$\left(\frac{D_a^2}{W_n^2}+2\xi\frac{D_a}{W_n}+1\right)\bar{R}_{ae}$$
$$= -\frac{\sum \bar{F}_A}{m_A W_n^2} + \frac{1}{W_n^2}(\bar{G}_{Ee} - \bar{G}_{EA} + \bar{G}_{Me} - \bar{G}_{MA}$$
$$+ \bar{G}_{Se} - \bar{G}_{SA} + \cdots)$$
$$- \frac{1}{W_n^2}[\bar{W}_{1A} \times (\bar{W}_{1A} \times \bar{R}_{Aa}) + (D_I\bar{W}_{1A}) \times \bar{R}_{Aa}$$
$$+ \bar{W}_{Ia} \times (\bar{W}_{Ia} \times \bar{R}_{ae}) + 2\bar{W}_{Ia} \times (D_a\bar{R}_{ae})$$
$$+ (D_I W_{Ia}) \times \bar{R}_{ae}] \qquad (179)$$

where

W_n = accelerometer undamped natural frequency $= \sqrt{k/Me}$,

ξ = accelerometer damping coefficient $= \dfrac{c}{2\sqrt{m_e k}}$.

The following observations may be made concerning (179):

1) The displacement vector $\bar{R}_{ae}$ represents the accelerometer output; in general, it is desired that

$$\bar{R}_{ae} \sim \sum \bar{F}_A/m_A;$$

that is, that the accelerometer measures the nonfield forces acting on the airframe.

2) In order that $\bar{R}_{ae} \sim \sum \bar{F}_A/m_A$, it is desirable that W_N be large. W_N large means that the instrument has good dynamic sensitivity; however, as can be seen from the right-hand side of (179), increasing W_N decreases the static sensitivity of the instrument.

3) Note that the terms in brackets are of two types, those containing $\bar{R}_{Aa}$, and those containing $\bar{R}_{ae}$. The terms containing $\bar{R}_{Aa}$ represent errors due to the fact that the accelerometer case is not located at the airframe center of mass. The $\bar{R}_{ae}$ terms represent errors due to the deflection of the mass element supports. Since

$$\bar{W}_{Ia} = \bar{W}I_A + \bar{W}_{Aa} = \bar{W}_{IA}$$

if follows that errors due to $\bar{R}_{ae}$ will be at least an order of magnitude smaller than these due to the accelerometer not being placed at the center of mass of the aircraft.

4) If the gravitational fields are considered to have negligible gradients over the volume of the airframe, then

$$\bar{G}_{Ee} + \bar{G}_{Me} + \bar{G}_{Se} + \cdots = \bar{G}_{EA} + \bar{G}_{MA} + \bar{G}_{SA} + \cdots;$$

that is, since

$$|\bar{G}_{EA}| \sim \frac{1}{R_{EA}^2}$$

$$|\bar{G}Ee| \sim \frac{1}{R_{Ee}^2}$$

and since $R_{Ee} = R_{EA} + r$, with $R_{EA} \gg r$, it follows that

$$|\bar{G}_{EA}| - |\bar{G}_{Ee}| \approx |\bar{G}_{EA} - \bar{G}_{Ee}| \approx 0.$$[53]

Therefore, neglecting the second-order effects, (179) reduces to

$$\bar{R}_{ae} \approx -\frac{\sum \bar{F}_A}{m_A W_n^2}. \qquad (179a)$$

It is convenient to relate the acceleration of the aircraft to a Newtonian frame centered at the earth, rather than the arbitrary set of inertial coordinates indicated by (179). To do this, note that

$$\bar{R}_{IA} = \bar{R}_{IE} + \bar{R}_{EA}. \qquad (180)$$

Eq. (170), when applied to the earth, yields

$$D_I^2\bar{R}_{IE} = \bar{G}_{ME} + \bar{G}_{SE} + \cdots. \qquad (181)$$

Since $\bar{W}_{I-I_E} = 0$ (that is, the Newtonian frame centered at the earth is irrational with respect to S_I), then

$$D_I^2\bar{R}_{EA} = D_{IE}^2\bar{R}_{EA}. \qquad (182)$$

Substituting (178) and (180)–(182) into (179) gives

$$\left[\left(\frac{D_a}{W_n}\right)^2 + 2\xi\left(\frac{D_a}{W_n}\right) + 1\right]\bar{R}_{ae}$$
$$= \frac{1}{W_n^2}[G_{EA} - D_{IE}^2\bar{R}_{EA} + (\bar{G}_{MA} - \bar{G}_{ME})$$
$$+ (\bar{G}_{SA} - \bar{G}_{SE}) + \cdots] \qquad (183)$$

where the terms in brackets in (179) have been dropped. According to Wrigley[54] the gravitational difference terms may be dropped, showing that the acceleration measures the effect of the earth's gravitational field on the airframe and the translational acceleration of the airframe with respect to a Newtonian reference at the earth's center.

XV. Conclusions

In this final part of a three-part series, we have extended the use of our vector algebra to two basic instruments, the gyroscope and the accelerometer. First the kinetic equations of a single-gimbal gyro were determined, using Newton's Second Law of motion as applied to a rotary system. Then the characteristics of two common two-gimbal gyros were examined. Finally, the dynamic behavior of an accelerometer was considered briefly, thus indicating how our vector algebra may be extended to translational as well as rotational systems, and to describe kinetic as well as kinematic relations.

[53] That is, since the vectors $\bar{G}_{EA}$ and $\bar{G}_{Ee}$ are virtually parallel, the difference between their magnitudes is approximately equal to the magnitude of their difference, etc.

[54] Wrigley, *et al.*, *op. cit.*, p. 37.

Statistical Evaluation of Digital-Analog Systems for Finite Operating Time*

R. B. NORTHROP† AND G. W. JOHNSON‡

Summary—**A frequency-domain mathematical model for comparing the performance of a linear sampled-data channel to a linear continuous channel is developed. The model assumes a suddenly applied stationary random input (input identically zero before the local time origin).**

The model allows an explicit algebraic definition of ensemble mean-squared error, after a finite operating time, by application of residue theory. The ideal or comparison channel need not be realizable.

The quasi-stationary characteristics of the excitation are accounted for by including a starting switch as a variable parameter within the appropriate system weighting functions.

The technique developed in this paper is utilized to evaluate the ensemble mean-squared error due to processing a vehicle velocity estimate (as might be supplied by an integrating accelerometer) in a digital computer. The velocity estimate is used to calculate a numerical value of present vehicle position. This problem is of fundamental importance in present-day pure inertial, pure Doppler, or inertial-Doppler navigation systems. The effect of quantization, inherent in the process of analog-to-digital conversion, may be included in this evaluation.

It is shown that for cases where system inputs can be approximated by narrow-band, first-order, Markoff-type power spectra, the practical engineering use of the simplest digital integration program (rectangular) or the simplest hold (box-car) is well justified. The examples illustrate the mathematical techniques necessary to compute the ensemble mean-squared error, and illustrate how several simplifying assumptions may be used.

I. Introduction

PRIOR to recent publications,[1,2] "frequency- domain" statistical studies required the assumption that the input characteristics be stationary for all time. Since all physical systems have an initiation time, this requires, from a practical standpoint, that the operating time be long compared to the system "memory" or settling time. However, in many important applications, the constraint that a system be in a steady-state operation imposes a severe limitation.

As pointed out by Johnson,[2] the response of an "infinite memory" filter (such as an integrating network) is in a transient state for all finite operating time, after sudden application of a stationary input. It is well known that the steady-state response of a linear, constant-parameter system due to stationary excitation is stationary. The fact that the transient response of linear invariant systems to suddenly applied stationary inputs is, in general, nonstationary is clearly demonstrated as follows.

The transient response of a stationary filter to a suddenly applied (or quasi-stationary) input is the response of an equivalent variable filter, including a starting switch, to an input which is assumed stationary for all time. Thus, it is only in the steady state, or after the variable effect of closing the starting switch is no longer detectable, that the characteristics of the response become stationary.

It is noted that statistical analysis procedures supply only a measure of average or mean properties of a signal. In general this refers to averaging some functional (usually the square) of a large number of signals representing a collection or "ensemble" of typical signals, subject to a given environment. Thus, the measure obtained, referred to as "ensemble average," is an estimate of the expected response characteristic and does not necessarily define a specific measure of any one response signal. However, for that subclass of stationary ensembles which obey the "ergodic" hypothesis an ensemble average is equivalent, in probability, to a corresponding average over time for any one signal or ensemble member. Thus, many publications concerning steady-state statistical analyses impose the constraint requiring inputs to be "ergodic" in order to use the time average of the squared response as a measure of system performance. Except for a few recent publications[3,4] all published statistical methods for sampled-data systems have required the "ergodic" constraint for application. It has been shown,[1] however, that the continuous response of a linear sampled data system is in general nonstationary, even in the steady state. However, the steady-state continuous response, of linear invariant sampled-data systems, is periodically stationary such that, if the inputs are ergodic, a time average over any one response is equivalent, in probability, to the mean value of all possible ensemble averages. It is noted that the time average obtained in this instance accounts only for the average properties of the intersample ripple in the response. In contrast, the ensemble average may vary markedly as a function of "phasing" of the output

* Manuscript received by the PGAC, April 13, 1959; revised manuscript received, January 12, 1960.
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[1] R. B. Northrop, "Statistical Model of Linear Sampled-Data Systems for Finite Operating Time," M.S. thesis, University of Connecticut, Storrs; 1958.

[2] G. W. Johnson, "Frequency-domain statistical model of linear variable networks for finite operating time," 1958 IRE NATIONAL CONVENTION RECORD, pt. 4, pp. 70–78.

[3] G. W. Johnson, "Statistical analysis of sampled-data systems," 1957 WESCON CONVENTION RECORD, pt. 4, pp. 187–195.

[4] B. Friedland, "Least Squares Filtering and Prediction of Nonstationary Sampled-Data System," presented at Conf. on Computers in Control Systems, Atlantic City, N. J.; October 16–18, 1957.

sampler, as will be noted with respect to a specific example later in this paper.

In summary, the model of this paper represents the most general statistical model for linear sampled-data systems. For instance, the steady-state performance is obtained by letting the operating time approach infinity, or, for ergodic inputs, a continuous time average may be obtained by taking the mean value of the nonstationary ensemble average over all possible phasings of time sampling.

II. Statistical Model for Finite Operating Time

The basic error generating model is shown in Fig. 1.

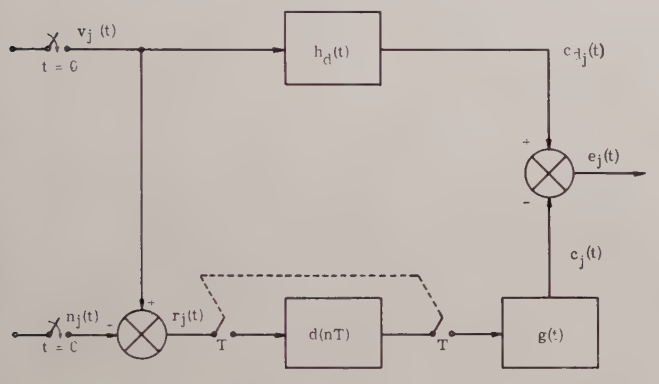

Fig. 1—Basic error generating model.

The response of a linear sampled-data channel is compared with the response of an ideal analog channel, for all finite operating time, after sudden excitation by an additive, stationary, random signal and noise.

It is assumed that the sampled-data channel of interest may be reduced to a pure digital filter in cascade with a continuous filter which in general will include both extrapolator-hold and analog dynamics.

The following nomenclature and definitions are pertinent to Fig. 1.

- $h_d(t)$ = weighting function of the fixed linear-continuous filter in the ideal or comparison channel (response at time t to a unit impulse applied at $t=0$). Not necessarily physically realizable.
- $g(t)$ = weighting function of a fixed linear continuous filter including extrapolator-hold dynamics. Assumed to be realizable.
- $d(nT)$ = number sequence weighting function of a fixed linear digital filter (digital filter numerical response at $t=nT$ to a numerical input of "1" at $t=0$). Assumed to be realizable.
- $r_j(t)$ = jth element of the sampled-data channel input ensemble.
- $v_j(t)$, $n_j(t)$ = jth elements of the signal and noise ensembles, respectively.
- $e_j(t)$ = jth element of the system error ensemble.
- $\{r_j(t)\} = \{v_j(t)+n_j(t)\}$ = stationary ensemble for $t\geq 0$.
- $\langle e_j^2(t)\rangle$ = ensemble mean-squared error at time t (averaged over all j).
- starting switch ($t=0$) = starting switch which is open for $t<0$ and closed for $t\geq 0$.
- sampling switch (T) = sampling switch which closes momentarily every T seconds and generates an impulse magnitude equal to the amplitude of the input signal at $t=nT$ (equivalent to an impulse modulator[5]).

With reference to Fig. 1, the ensemble mean-squared error may be expressed as

$$\langle e_j^2(t)\rangle = \langle c_{d_j}{}^2(t)\rangle - 2\langle c_{d_j}(t)c_j(t)\rangle + \langle c_j{}^2(t)\rangle. \qquad (1)$$

It is shown in Appendix I that the three terms on the right-hand side of (1) may be expressed as equivalent complex integrals as follows:

$$\langle e_j^2(t)\rangle = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Phi_{vv}(s)H_d'(s,t)H_d'(-s,t)ds \qquad (2a)$$

$$-\frac{1}{\pi j}\int_{-j\infty}^{j\infty}[\Phi_{vr}(s)H_d'(-s,t)e^{-smT}]F'^*(e^{sT},m,p)ds \qquad (2b)$$

$$+\frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\Phi_{rr}(s)F'^*(e^{sT},m,p)F'^*(e^{-sT},m,p)ds \qquad (2c)$$

where $\Phi_{vv}(s)$ and $\Phi_{rr}(s)$ are the spectral densities of signal and input, respectively, and $\Phi_{vr}(s)$ is the cross spectral density between signal and input. Spectral density is defined as follows:

$$\Phi_{xy}(s) = \int_{-\infty}^{\infty}\phi_{xy}(\tau)e^{-s\tau}d\tau \qquad (3)$$

where $\phi_{xy}(\tau)$ is the correlation function between $\{x_j(t)\}$ and $\{y_j(t)\}$ defined as

$$\phi_{xy}(\tau) \equiv \langle x_j(t)y_j(t+\tau)\rangle \qquad \text{for } t\geq 0, \qquad (4)$$

where $\{x_j(t)\}$ and $\{y_j(t+\tau)\}$ are stationary for $t\geq 0$.

The terms $H_d'(s,t)$ and $F'^*(e^{sT},m,p)$ are the equivalent transfer functions of the ideal and sampled-data channels, respectively, including the effect of a starting switch as a variable parameter, where

$$H_d'(s,t) \triangleq H_d(s) - \frac{1}{2\pi j}\oint_{\Gamma_d} H_d(w)\frac{e^{-(s-w)t}}{s-w}dw. \qquad (5)$$

If $H_d(s)$ is realizable, Γ_d encloses the poles of $H_d(w)$ but excludes $w=s$. In the event that the comparison channel is not physically realizable, Γ_d is a closed contour taken in the positive sense enclosing all the poles to the left of the analytic strip for $H_d(s)$ (the poles which define the impulse response for positive time).

In general, the bilateral Laplace transform is used since $h_d(t)$ need not be physically realizable.

[5] W. K. Linvill, "Sampled-data control systems studies through comparison of sampling with amplitude modulation," *Trans. AIEE*, pt. II, vol. 70, pp. 1779–1782; 1951.

Before defining $F'^*(e^{sT}, m, p)$ it is convenient to express the latter two integrals in (2) in terms of the z-transform variable. This development is shown in Appendix I and results in

$$\langle e_j^2(t)\rangle = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Phi_{vv}(s)H_d'(s,t)H_d'(-s,t)ds \quad (6a)$$

$$-\frac{1}{\pi j}\int_I Z[\Phi_{vr}(s)H_d'(-s,t)e^{-smT}]F'^*(z,m,p)\frac{dz}{z} \quad (6b)$$

$$+\frac{1}{2\pi j}\int_I \Phi_{rr}^*(z)F'^*(z,m,p)F'^*\left(\frac{1}{z}, m, p\right)\frac{dz}{z} \quad (6c)$$

where I is the "semi-open" line segment $|z|=1$, $-\pi+A\leq \arg z<\pi+A$, taken in a counter-clockwise sense and A is an arbitrary real number. The procedure for evaluating the integrals of (6) by integrating over equivalent closed contours will be illustrated in the next section with respect to specific examples.

With reference to (6), the notation

$$Z[\Phi_{vr}(s)H_d'(-s,t)e^{-smT}]$$

denotes the bilateral z transform of the quantity in the brackets, defined in terms of the "Poisson" sum as

$$Z[\Phi_{vr}(s)H_d'(-s,t)e^{-smT}] \triangleq \frac{1}{T}\sum_{n=-\infty}^{n=\infty} \Phi_{vr}\left(s+jn\frac{2\pi}{T}\right) H_d'\left(-s-jn\frac{2\pi}{T}, t\right)e^{-(s+jn(2\pi/T))mT}. \quad (7)$$

The term $\Phi_{rr}^*(z)$ represents the sampled power spectral density defined in terms of the Poisson sum as

$$\Phi_{rr}^*(z) \triangleq \frac{1}{T}\sum_{n=-\infty}^{n=\infty}\Phi_{rr}\left(s+jn\frac{2\pi}{T}\right). \quad (8)$$

The term $F'^*(z, m, p)$ represents the truncated, modified z transform of the over-all sampled-data channel, where

$$F^*(z,m) = D^*(z)G^*(z,m) \quad (9)$$

and

$$D^*(z) = \sum_{k=0}^{k=\infty} d(kT)z^{-k} \quad (10)$$

and

$$G^*(z,m) = \sum_{n=0}^{n=\infty} g(nT+mT)z^{-n} \quad (11)$$

where m ranges $0\leq m<1$. It is noted that this definition of the modified z transform differs from that given by Barker[6] and Jury[7] by a factor of z^{-1}. The authors prefer this definition since $G^*(z, m)$ approaches $G^*(z)$ as m approaches zero.

$F'^*(z, m, p)$ is obtained by truncating $F^*(z, m)$ by means of complex convolution in z.

$$F'^*(z,m,p) = F^*(z,m) - \frac{1}{2\pi j}\cdot\int_{\Gamma_y}\frac{F^*(y,m)(z/y)^{-p}}{(z/y-1)}\frac{dy}{y}. \quad (12)$$

Γ_y encloses the poles of $F^*(y, m)$ but excludes the point $y=z$, and p is an integer such that $t=pT+mT$, where $0\leq m<1$, and $t\geq 0$.

The expression for ensemble mean-squared error as given in (6) will be used in the next section to evaluate the relative merits of various digital integration programs and extrapolator-hold filters.

III. Some Illustrative Examples

As stated in the Introduction, the frequency domain statistical model presented in this paper is necessary to evaluate the performance of sampled-data systems when the system settling time is long compared to the operating time [time between initial application of the stationary random input and evaluation of the ensemble mean-squared error (emse)].

An example of such a system, of general interest, is any of the commonly-used digital integration programs, followed by some extrapolation-hold dynamics. Conventional steady-state mean-squared error (mse) analysis techniques would, for the case of infinite-memory digital integration programs, give rise to an unbounded mse. (The inversion integrals are not defined.)

The techniques of this paper, as applied to the examples involving digital integration below, allows evaluation of the emse for all $t\geq 0$, and not just at the sampling instants. It is seen that the steady-state, or infinite operating time emse is indeed unbounded for these examples.

A practical application of digital integration programs is in the inertial guidance field. For example, a digital velocity measure can be integrated to obtain an estimate of present vehicle position. In this case, it is evident that system operating time must be considered to be finite.

It is assumed that a measure of the error due to numerical integration of a random velocity signal is of interest. It is also assumed that the guidance interval is sufficiently large compared to the numerical computational period of the integration program for the intersampling behavior of the esme to be negligible, compared to the total emse level. Thus, only the sampling instants will be considered in the first two examples. If relatively short operating times are considered, where the over-all level of the emse has not had time to build up, the intersampling behavior may not be negligible, and may be found by letting m vary, $0\leq m<1$.

The random velocity input for the following examples is considered to be a relatively narrow-band signal,

[6] R. H. Barker, "The pulse transfer function and its application to sampling servo systems" *Proc. IEE*, vol. 99, pt. IV, pp. 302–317; December, 1952.

[7] E. I. Jury, "Discussion of G. W. Johnson, D. P. Lindorff, and C. G. A. Nordling on extension of continuous-data system design techniques to sampled-data control systems," *Trans AIEE (Applications and Industry)*, vol. 74, pt. II, pp. 252–263; September 1955.

compared to the computational frequency, with a power spectral density of the form

$$\Phi_{vv}(\omega) = \frac{2a\sigma_v^2}{a^2 + \omega^2} \cdot \tag{13}$$

Using this input signal, a comparison is made between a general second-order digital program, rectangular integration, and ideal integration (1/s). The digital to analog conversion is accomplished by a simple "box-car"-hold filter. Computational delay is neglected in the examples (see Fig. 2).

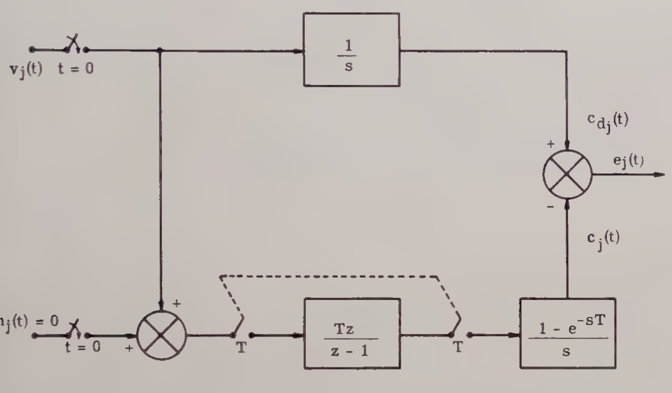

Fig. 2—Error-generating model for the first example.

Example 1

The first step in computing the emse for the model involving rectangular integration is to determine the truncated, modified, transfer functions of the comparison and sampled-data channels. From (5),

$$H_d'(s, t) = \frac{1}{s} - \frac{1}{2\pi j}\int_{\Gamma_w} \frac{1}{w}\,\frac{e^{-(s-w)t}}{s-w}\,dw = \frac{1-e^{-st}}{s} \tag{14}$$

and

$$H_d'(-s, t) = \frac{e^{st} - 1}{s} \cdot \tag{15}$$

In general, the truncation process converts the Laplace transform of a realizable filter weighting function to an entire transcendental function (analytic in the finite s plane). If $H_d(s)$ is the Laplace transform of a two-sided time function, then the truncation process converts the left-half plane portion of $H_d(s)$ to an entire transcendental function, analytic in the left-half s plane.

For the sampled data channel,

$$F^*(z, m) = D^*(z)G^*(z, m) = \frac{Tz}{z-1} \times 1. \tag{16}$$

Truncating, by use of (12),

$$F'^*(z, m, p) = \frac{Tz}{z-1} - \frac{1}{2\pi j}\int_I \frac{Ty}{y-1}\,\frac{(z/y)^{-p}}{(z/y-1)}\,\frac{dy}{y}$$
$$= \frac{T(z - z^{-p})}{z-1},\ p \geq 0 \tag{17}$$

and

$$F'^*\left(\frac{1}{z}, m, p\right) = \frac{T(z^{-1} - z^p)}{(z^{-1} - 1)},\ p \geq 0. \tag{18}$$

The emse may now be written using the three integrals of (6). Since the large-time behavior at sampling instants is of interest in this case, $t = pT + mT$ reduces to $t = pT$, where p is a positive integer. Thus,

$$\frac{\langle e_j^2(t)\rangle}{\sigma_v^2}$$
$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \frac{2a}{(a^2 - s^2)}\,\frac{(e^{spT}-1)}{s}\,\frac{(1-e^{-spT})}{s}\,ds \tag{19a}$$
$$- \frac{1}{\pi j}\int_I Z\left[\frac{2a}{(a^2-s^2)}\,\frac{(e^{spT}-1)}{s}\right]\frac{T(z-z^{-p})}{(z-1)}\,\frac{dz}{z} \tag{19b}$$
$$+ \frac{T^2}{2\pi j}\int_I \frac{(z-z^{-p})}{(z-1)}\,\frac{(z^{-1}-z^p)}{(z^{-1}-1)}\cdot\frac{(-2z\sinh x)}{(z-e^x)(z-e^{-x})}\,\frac{dz}{z} \tag{19c}$$

where $x = aT$.

The integrand of (19a) is seen to have finite s-plane poles at $s = +a$, and is analytic at $s = 0$. The path of integration may be modified to avoid the point $s = 0$ by an infinitesimal semicircular arc to either the right or left, and the value of (19a) will be unaltered. In this case, the origin will be passed to the right, defining the contour L (see Fig. 3).

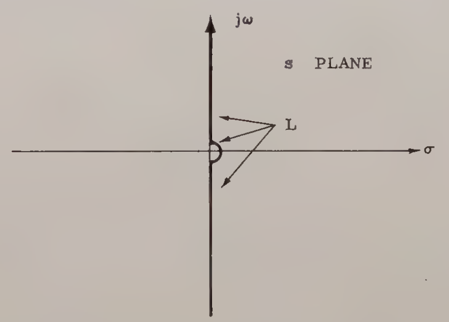

Fig. 3—Modified (equivalent) path of integration for (21a).

The integral (19a) on the modified contour L may be expressed as the sum of two integrals:

$$I_1 = \frac{2a}{2\pi j}\int_L \frac{e^{spT} - 2}{s^2(a+s)(a-s)}\,ds$$
$$+ \frac{2a}{2\pi j}\int_L \frac{e^{-spT}}{s^2(a+s)(a-s)}\,ds. \tag{20}$$

The value of the integral for all $t \geq 0$ is of interest. Eq. (20) may be evaluated by means of residue theory; the modified path of integration, L, may be closed by a semicircular arc of infinite radius for either integral. The direction of this arc depends on the sign of the exponential term in the numerator, and must be chosen so that the integral along the arc of closure adds nothing to the value of (20). It is seen that the left-hand integral in (20)

must be closed to the left and the right-hand integral to the right. Application of residue theory yields

$$I_1 = \frac{2pT}{a} + \frac{2}{a^2}(e^{-apT} - 1). \tag{21}$$

The cross-term of (19b) requires the evaluation of the bilateral z transform of the bracketed quantity. The easiest way to do this is to write the bracketed quantity in terms of its inverse Laplace transform and use the definition

$$Z[H(s)] = Z[h(t)] = \sum_{n=-\infty}^{n=\infty} h(nT)z^{-n} \tag{22}$$

where

$$h(t) = \mathcal{L}^{-1}[H(s)].$$

Thus, the cross-term may be written

$$I_2 = \frac{-2}{2\pi j}\int_I \frac{2(\cosh x - 1)[z^2 + z - z^p(z^2 + z)]T(z - z^{-p})}{a(z-1)(z-e^{-x})(z-e^{x})(z-1)} \cdot \frac{dz}{z}. \tag{23}$$

The integrand of (23) is seen to be analytic at $z=+1$, for both factors, as a result of the truncation process. Since there is no pole at $z=+1$, the semi-open line segment, I, may be modified to avoid the point $z=+1$ by an infinitesimal semicircular arc. The arc will be chosen to pass to the right in this case. The modified counter, I', is shown in Fig. 4.

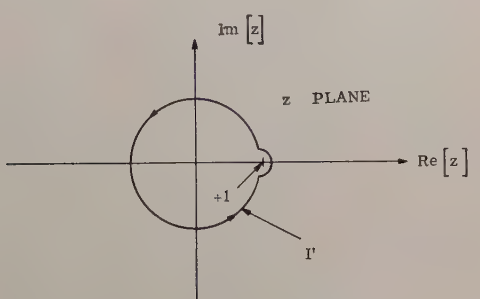

Fig. 4—Modification of the semi-open line segment, I.

Eq. (23) may now be written as a sum of two integrals on the modified contour, I'.

$$I_2 = \frac{4T(\cosh x - 1)}{a} \cdot \left\{\frac{1}{2\pi j}\int_{I'} \frac{z^{p+1} + z^{p+2} - (z^2 + 2z + 1)}{(z-1)^2(z-e^{-x})(z-e^{x})} dz + \frac{1}{2\pi j}\int_{I'} \frac{z+1}{z^p(z-1)^2(z-e^{-x})(z-e^{x})} dz\right\}. \tag{24}$$

As shown in Appendix II, the modified semi-open line segment I' may be closed either to make a contour-clockwise enclosure of the region $0<|z|<1$, or a clockwise enclosure of the region $1<|z|<\infty$, dependent on the exponent of z, in order that the values of (26) are unaltered. Thus, the first integral in (24) encloses the region $0<|z|<1$, and the second integral encloses $1<|z|<\infty$. Application of residue theory yields

$$I_2 = \frac{T^2}{x}\left\{-4p - 2 + \frac{2(\cosh x - 1)[(e^{x}+1)^2 - (e^{-x}-1)e^{-x(p+1)}]}{\sinh x(e^{-x}-1)^2} - \frac{e^{-x(p+1)}(e^{2x}+e^{x})2(\cosh x - 1)}{\sinh x(e^{x}-1)^2}\right\}. \tag{25}$$

Finally, the third integral (19c) may be evaluated in a similar fashion as (23). The result is

$$I_3 = T^2\left\{\frac{e^{-x}(e^{-x(p+1)} - 2)}{(e^{-x}-1)^2} + \frac{\sinh x(p+1)}{\cosh x - 1} + \frac{e^{-xp}}{(e^{x}-1)^2}\right\}. \tag{26}$$

The ensemble mean-squared error for the case of a suddenly-applied stationary random input signal with a power spectrum of the form of (13) is the sum of (21), (25), and (26). This emse is valid only for sampling instants where the intersampling behavior of the emse is negligible compared with the over-all emse level. For values of p large enough to cause $xp>15$, terms containing e^{-xp} and $e^{-x(p+1)}$ are negligibly small, and may be dropped. Thus, the emse for the model of Fig. 2, valid for $t=pT$ where $xp>15$, is given by

$$\frac{\langle e^2(pt)\rangle}{\sigma_j^2} = T^2p\left\{\frac{\sinh x}{\cosh x - 1} - \frac{2}{x}\right\} + T^2\left\{\left(\frac{\sinh x}{\cosh x - 1} + \frac{2}{x}\right) + \frac{2(\cosh x - 1)(e^{-x}+1)^2}{x\sinh x(e^{-x}-1)^2} - \left(\frac{2}{x^2} + \frac{2e^{-x}}{(e^{-x}-1)^2}\right)\right\}. \tag{27}$$

It is seen that (27) is linear in p.

Example 2—A General Second-Order Integration Program

A logical next step is to consider more sophisticated schemes of digital integration to see if a program exists that will give least emse at a given large operating time.

Fig. 5 illustrates such a general second-order approximation to $1/s$. This program is a so-called "oscillating program." That is, in the steady state, the number sequence weighting function is

$$c(nT) = \frac{2T}{r+2}(2) \qquad \text{for even } n \geq 2$$

$$c(nT) = \frac{2T}{r+2}(r) \qquad \text{for odd } n \geq 1. \tag{28}$$

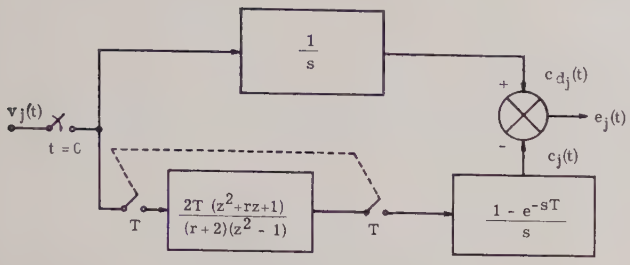

Fig. 5—Error-generating model for example 2.

The constant $2T/r+2$ is chosen to cause the weighting function to oscillate about an average steady-state value of 1, given an input of "1" at $t=0$. If $r=2$, the general program reduces to trapezoidal integration. If $r=4$, the program becomes the well-known Simpson's 1/3 rule. Other values of r are presumed to be of interest.

Following the methods outlined in the first example, the emse for this system has been calculated and found to be, for $xp>15$,

$$\frac{\langle e^2(pt)\rangle}{\sigma_v^2} = T^2 p\left\{\frac{\sinh x}{\cosh x - 1} - \frac{2}{x} + \left(\frac{r-2}{r+2}\right)^2 \times \frac{\sinh x}{\cosh x - 1}\right\} + b_2(r). \quad (29)$$

$b_2(r)$ is also a function of x and r, and is given by

$$b_2(r) = T^2\left\{\frac{2\sinh x}{(\cosh x - 1)(r+2)} + \frac{2(2-r)\sinh x}{(r+2)^2(\cosh x + 1)} + \frac{4(\cosh x - 1)\left[e^{-2x} + 2re^{-x} + 3\right]}{(r+2)\,x\,(\sinh x)(e^{-x}-1)^2} - \frac{8}{(r+2)x} - \frac{4}{(r+2)^2}\left[\frac{e^{-4x} + 6e^{-2x} + 1 + 2r(2e^{-2x} + re^{-x} + 2)e^{-x}}{(e^{-x}+1)^2(e^{-x}-1)^2}\right] - \frac{2}{x^2}\right\} \quad (30)$$

(valid for p even)

or

$$b_2(r) = T^2\left\{\frac{r\sinh x}{(\cosh x - 1)(r+2)} + \frac{r(r-2)\sinh x}{(r+2)^2(\cosh x + 1)} + \frac{4(\cosh x - 1)\left[(e^{-x}+1)(e^{-x}+r+1)\right]}{(r+2)\,x\,(\sinh x)(e^{-x}-1)^2} - \frac{2}{x^2} - \frac{4r}{(r+2)x} - \frac{4}{(r+2)^2}\left[\frac{e^{-4x} + 6e^{-2x} + 1 + 2r(2e^{-2x} + re^{-x} + 2)e^{-x}}{(e^{-x}-1)^2(e^{-x}+1)^2}\right]\right\} \quad (31)$$

(valid for p odd).

It is evident that the rate of increase of emse given by the p coefficient in (29) is a minimum when $r=2$. For this case, the general second-order program reduces to trapezoidal integration. Moreover, it is noted that the rate of increase of emse for the trapezoidal integrator is the same as for the rectangular integrator treated in the first example.

The rather formidable constant given by (30) and (31) is found to be of the same order of magnitude as the constant term for rectangular integration, for ranges of the variables $0<x<1$, $0\leq r\leq 10$. The p coefficient then is the important factor when large operating times are considered.

Thus, the logical choice of integration program would be the rectangular one, using the least emse criterion. Rectangular integration is certainly the easiest to implement, and it gives performance that is as good, if not better, than a more complex second-order program, for large operating times and the input specified.

For short operating times ($xp<15$), this will not necessarily be so, and a comparison would have to be made that would include exact behavior of the constant terms, and possibly the emse at intersampling instants.

Example 3—Comparison of Basic Extrapolator—Hold Filters

In the previous two examples, behavior of the emse at sampling instants only was considered, for large system operating times. In this example, the extrapolator-hold filters involved have weighting functions that are zero after at least two sampling periods, as compared to the infinite settling times involved in the first two examples. Because of this finite settling time, the error-generating models shown below reach "steady-state" for $t\geq 2T$ (after initial application of the stationary random input).

It is seen that a great simplication results if values of $p\geq 2$ are considered. The truncation process is no longer necessary, since the weighting functions of the extrapolator-hold filters are identically zero for $t>2T$ (see Fig. 6).

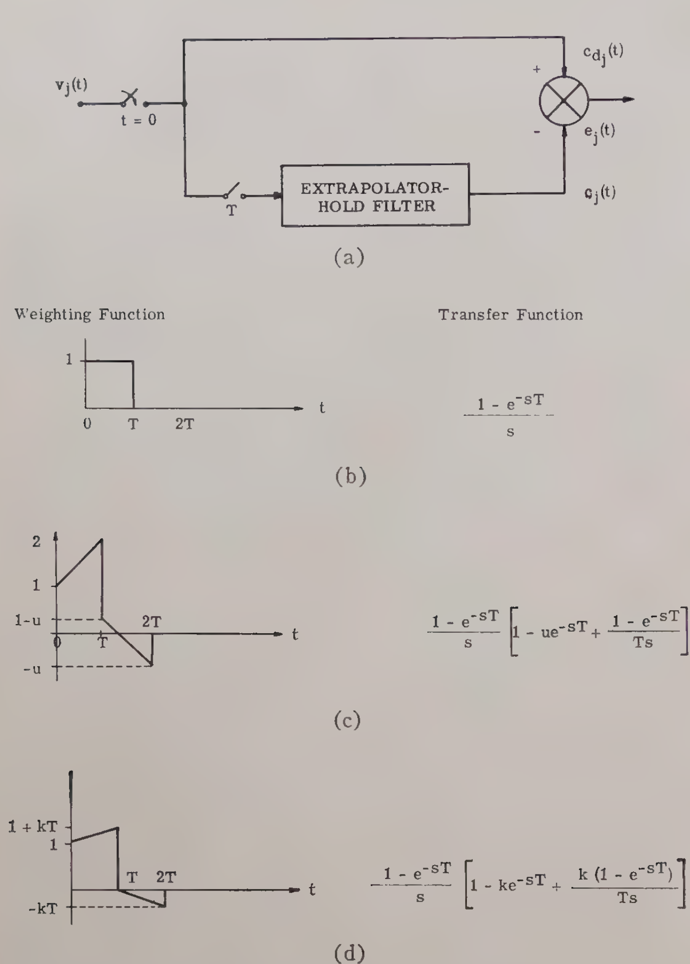

Fig. 6—(a) Basic error-generating model for example 3. (b) Zero-order or box-car hold. (c) First-order hold with variable reset. (d) First-order hold with variable velocity correction.[8]

The input for these three examples is a random signal with a power spectral density given by (13).

1) Considering the zero-order hold filter first, application of (6a), (6b), and (6e) yields

$$\langle e^2(t)\rangle = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty}\frac{2a\sigma_v^2}{a^2-s^2}ds \tag{32a}$$

$$-\frac{1}{\pi j}\int_I Z\left[\frac{2a\sigma v^2}{a^2-s^2}e^{-smT}\right]\frac{dz}{z} \tag{32b}$$

$$+\frac{1}{2\pi j}\int_I Z\left[\frac{2a\sigma v^2}{a^2-s^2}\right]\frac{dz}{z}. \tag{32c}$$

The first and third integrals are equal to σ_v^2. z-transforming the bracketed quantity, the second integral may be written

$$I_2 = \frac{1}{2\pi j}\int_I \frac{4\sigma_v^2[z\sinh(x-xm)+\sinh(mx)]}{(z-e^{-x})(z-e^{x})z}dz. \tag{33}$$

The final result, after algebraic simplification, is

$$\frac{\langle e^2(t)\rangle}{\sigma_v^2} = 2(1-e^{-mx}) \qquad \text{valid for } t \geq T. \tag{34}$$

2) The emse for the first-order hold with variable reset is found to be

$$\frac{\langle e^2(t)\rangle}{\sigma_v^2} = 1 - 2e^{-mx}[m+1+(1-m-u)e^{-x}]$$
$$+(m+1)^2+(u-1)^2+m(m-2+2u)$$
$$+2e^{-x}(1-u-mu-m^2)$$
$$(\text{valid for } t \geq 2T). \tag{35}$$

3) The emse for the first-order hold with variable velocity correction is

$$\frac{\langle e^2(t)\rangle}{\sigma_v^2} = 2(1-e^{-mx})+2(1-e^{-x})$$
$$\cdot[k^2m^2+km(1-e^{-mx})]$$
$$(\text{valid for } t \geq 2T). \tag{36}$$

It is noted that at sampling instants ($m=0$), the emse for all three holds is zero.

Conventional mean-squared-error techniques give the average value of the steady-state mse over m. In the examples above,

$$\widetilde{e^2} = \frac{1}{1-0}\int_0^1 \langle e^2(t)\rangle dm, \tag{37}$$

the emse is given for all m, $0 \leq m < 1$, and use of (37) allows calculation of the average mse with respect to m, if so desired.

A numerical comparison has been made between the holds for various values of m and x. Values of u_{opt} and k_{opt} have been derived to minimize the emse at a given x and m.

$$u_{\text{opt}} = 1 - e^{-x(m+1)} + (m+1)e^{-x} - m \tag{38}$$

$$k_{\text{opt}} = -\frac{1-e^{-mx}}{2m}. \tag{39}$$

Figs. 7 and 8 illustrate the optimum behavior of the emse for various representative operating conditions. It is at once evident that the best choice of hold is the simple box-car. The hold of Fig. 6(d) gives almost identical performance, but of course is harder to implement. The choice of the box-car is based on least emse for any values of m and x in the ranges $0 \leq m < 1$, $0 < x \leq 2$. For certain deterministic inputs, and other types of input power spectra, the box-car hold filter is not necessarily "optimum." It is felt, however, that the example is significant, since a large class of random input signals are characterized by spectra of the form of (13) or may be approximated by it.

IV. Conclusion

Using the least emse criterion as a measure of system performance, the preceding examples have shown that

[8] After J. R. Raggazzini and G. F. Franklin, "Sampled-Data Control Systems," McGraw-Hill Book Co., Inc., New York, N. Y., pp. 40–42; 1958.

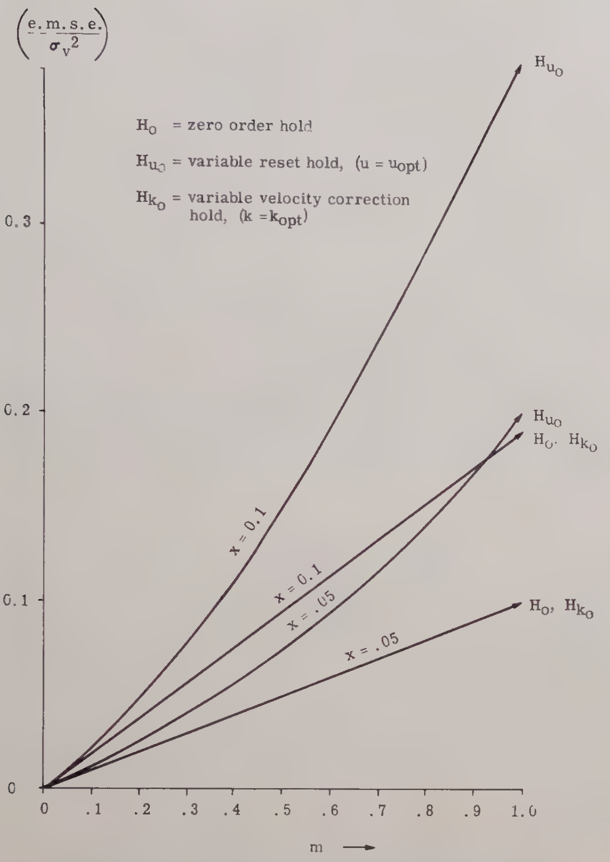

Fig. 7—Growth of emse with intersample time.

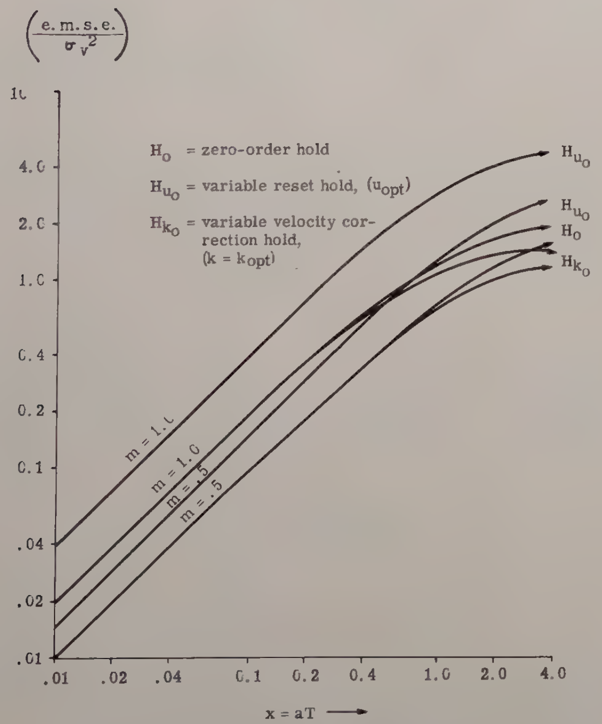

Fig. 8—Variation of emse with x.

for cases where system inputs can be approximated by narrow-band, first-order, Markoff-type spectra, the practical engineering use of the simplest digital integrator (rectangular) or the simplest hold (box-car) is well justified.

It must be remembered that the conditions specified for the examples constrain the generality of the results, and good judgment must be used not to "over-extrapolate" the significance of the conclusions set forth above.

The analytical technique described in this paper makes it possible to evaluate the performance of linear sampled-data systems subjected to suddenly applied random signals, stationary after the local time of origin and identically zero before. Using algebraic means, it is possible to evaluate system emse as a function of system operating time, and not just at the sampling instants. Use of the technique is indicated where system settling time is long compared to the operating time, *i.e.*, when the system cannot be considered to be in the steady state.

The technique utilizes readily available tables of "modified" z transforms and the application of elementary residue theory for the evaluation of the complex integrals involved. The artifice of truncation is employed in order that certain time-domain integral expressions concerned in the derivation of the model may be expressed in the frequency domain in closed form. Truncating a fixed, linear filter weighting function converts the filter to a special type of variable filter, whose weighting function is required to be zero for values of the age variable greater than the operating time of the system. The technique of truncation is readily carried over into the frequency domain, using complex convolution, for both analog[2] and sampled-data filters.[1]

While the frequency-domain statistical model presented in this paper serves as a basis for wide areas of investigation and design in terms of digital integration and hold dynamics, it should be noted that for other cases, where the systems considered do not have infinite memory, judgment must be used to ascertain whether the transient statistical model is necessary. If the system operating time is long compared to the system settling time, conventional steady-state mse evaluation techniques may be justified in terms of their relative algebraic simplicity.

The model of this paper is especially useful for evaluation of digital integration programs, followed by extrapolator-hold dynamics. Sampled-data systems of this kind are a vital part of many present-day inertial navigation systems, where optimum performance is tantamount to operational success. The use of the model of this paper allows evaluation of such systems in terms of the emse at a finite operating time. Heretofore, this has not been possible, since conventional steady-state mean-squared error techniques give rise to an unbounded result.

Appendix I

Derivation of the Frequency-Domain Statistical Model

The assumptions stated in Section II pertaining to the error-generating model of Fig. 1 are assumed valid for the following development.

The digital filter $D^*(z)$ is assumed to operate on purely numerical data. Hence its dynamics may readily be described in terms of a rational function of the dimensionless complex variable z.

The sampled data system dynamics following the digital filter $g(t)$ (a digital-to-analog converter, for example) are assumed to respond only to the amplitudes of the digital filter periodic output pulses. Hence, these analog dynamics may be considered to be excited by periodic impulses with areas equal to the corresponding periodic numerical output of the digital filter. Thus, no weighting constant need be used to account for finite pulse width, and all samplers may be considered to be equivalent to ideal "impulse modulators."

Considering the jth (typical) members of the signal, noise, and error ensembles, the output of the error-generating model may be written in terms of the pertinent weighting functions of the error-generating model. Subject to the conditions imposed on the convolution processes by realizability and the switch closing at $t=0$, the error at a time $t\geq 0$ is

$$e_i(t) = \int_{-\infty}^{t} h_d(u)v_i(t-u)du - \sum_{n=0}^{p} r_i(nT)f(t-nT). \quad (40)$$

The second term in (40) is merely a summation of impulse responses of the sampled-data channel to a periodically sampled ensemble member, $r_i(nT)$. The upper limit on the summation, p, is defined so that $pT\leq t\leq (p+1)P$.

Since the expected value of the squared error is of interest, (40) may be squared. Assuming a dominated convergence, the various terms may be rearranged, and the expectation of the squared error ensemble (at a time $t\geq 0$) taken. Thus, the ensemble mean-squared error may be written

$$\langle e_i^2(t)\rangle = \int_{-\infty}^{t}\int_{-\infty}^{t} h_d(u)h_d(\zeta)\langle v_i(t-u)v_i(t-\zeta)\rangle du d\zeta \quad (41a)$$

$$- 2\int_{-\infty}^{t} h_d(u)du \sum_{n=0}^{p} f(t-nT)\langle v_i(t-u)r_i(nT)\rangle \quad (41b)$$

$$+ \sum_{n=0}^{p} f(t-nT)\sum_{k=0}^{p} f(t-kT)\langle r_i(nT)r_i(kT)\rangle. \quad (41c)$$

The brackets $\langle \cdots \rangle$ denote the operation of averaging over all ensemble members at a time $t\geq 0$. Note that no assumptions concerning ergodicity have been made. Ensemble averages must be used because the squared error ensemble is, in general, a nonstationary process for all $t\geq 0$.

The expected value of the product of two variables, $x(u)y(y)$, is conventionally defined as the cross-correlation function of the variables. If $x=y$,

$$E\{x(u)y(\zeta)\} \triangleq \langle x_i(u)y_i(\zeta)\rangle = \phi_{xy}(u,\zeta) \quad (42)$$

is simply the autocorrelation function of x. When $\{x\}$ and $\{y\}$ are stationary random processes, the correlation function of (43) is a function of the time differences only. Since $\{v\}$ is a stationary ensemble for $t\geq 0$, (41a) may be written

$$I_1 = \int_{-\infty}^{t}\int_{-\infty}^{t} h_d(u)h_d(\zeta)\phi_{vv}(u-\zeta)du d\zeta. \quad (43)$$

The upper limits of the integrals in (43) may be extended to infinity if the fixed filter weighting function is modified and considered as a special class of variable filter so that

$$\begin{aligned} h_d'(t,u) &\triangleq h_d(u), && \text{for } u\leq t \\ h_d'(t,u) &\triangleq 0, && \text{for } u > t. \end{aligned} \quad (44)$$

This may be expressed mathematically by use of the unit step function

$$\begin{aligned} U(u-t) &\triangleq 1, && \text{for } u > t \\ U(u-t) &\triangleq 0, && \text{for } u\leq t. \end{aligned} \quad (45)$$

Thus

$$h_d'(t,u) = h_d(u)[1-U(u-t)]. \quad (46)$$

This important modification technique is known as truncation.[1] Now (43) may be written

$$I_1 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h_d'(t,u)h_d'(t,\zeta)\phi_{vv}(u-\zeta)du d\zeta. \quad (47)$$

Eq. (47) may be written in the frequency domain using the uniqueness of the bilateral Laplace transform. $\phi_{vv}(u-\zeta)$ may be written in terms of the inverse transform of its power spectral density function. Thus

$$\phi_{vv}(u-\zeta) = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Phi_{vv}(s)e^{s(u-\zeta)}ds, \quad (48)$$

(48) may be substituted into (47), and, assuming a dominated convergence, we may rearrange integral signs and regroup terms to obtain

$$I_1 = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Phi_{vv}(s)\int_{-\infty}^{\infty} h_d'(t,u)e^{su}du \cdot \int_{-\infty}^{\infty} h_d'(t,\zeta)e^{-s\zeta}d\zeta ds. \quad (49)$$

It is seen at once that the two real integrals in (49) are simply bilateral Laplace transforms of the modified comparison channel filter weighting function.

Eq. (49) may be written in the frequency domain.[9]

[9] Equivalent to the development given by Johnson, *op. cit.*, footnote 2.

$$I_1 = \langle c_{di}^2(t) \rangle = \frac{1}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{vv}(s) H_d'(s, t) H_d'(-s, t) ds. \quad (50)$$

$H_d'(s, t)$ may be calculated from (46) using complex convolution. Since $h_d(t)$ is, in general, the weighting function of a nonrealizable filter, it is assumed that $H_d(s)$ may be described by a bilateral Laplace transformation. $H_d(s)$ is therefore defined for the $R_e(s)$ lying in a strip in the s plane which, from a practical constraint, includes the $s = jw$ axis. Thus $H_d'(s, t)$ may be expressed.

$$H_d'(s, t) = H_d(s) - \frac{1}{2\pi j} \oint_{\Gamma_w} H_d(w) \frac{e^{-(s-w)t}}{s - w} dw \quad (51)$$

where Γ_w is a contour enclosing the poles to the left of the analytic strip for $H_d(s)$. In the event that $H_d(s)$ is realizable, Γ_w encloses all the poles of $H_d(w)$ excluding $w = s$.

The next step in writing the ensemble mean-squared error as a function of system operating time is to consider (41b). The upper limit on the integral sign may be extended to infinity following (44)–(47). Since $\{v\}$ is a stationary ensemble for $t \geq 0$, and $\{r(nT)\}$ is a stationary random number sequence ensemble for $t = nT \geq 0$ (where n is an integer), the correlation function involving $v_i(t-u)$ and $r_i(nT)$ is stationary for $u \leq t$ and $n \geq 0$. Thus (41b) may be written

$$I_2 = -2 \int_{-\infty}^{\infty} h_d'(t, u) du \sum_{h=0}^{p} f(t - nT) \phi_{vr}(nT - t + u)$$
$$u \leq t, \, n \geq 0. \quad (52)$$

The correlation function in (52) may be written in terms of its inverse Laplace transform. This may be substituted into (52), similar to the operation on (47). Thus,

$$I_2 = -\frac{2}{2\pi j} \int_{-j\infty}^{j\infty} \Phi_{vr}(s) H_d'(-s, t) e^{-st}$$
$$\sum_{n=0}^{p} f(t - nT) e^{snT} ds. \quad (53)$$

The summation in (53) may be written in closed form through an extension of the truncation technique shown in (44)–(46). Since in general, t may lie at or between sampling instants, the substitution $t = pT + mT$ is made, where

$$\begin{Bmatrix} 0 \leq pT \leq t < (p+1)T \\ 0 \leq m < 1 \end{Bmatrix}.$$

A further substitution in the argument of $f(t - nT)$, $l = p - n$ is made. Thus

$$\sum_{l=0}^{l=p} f(mT + lT) e^{-slT} e^{spT}. \quad (54)$$

Note that the e^{spT} factor cancels when (54) is substituted back into (53); it will therefore be neglected in the rest of this development. The upper limit of summation in (54) may be extended to ∞ if the weighting function is modified by the truncation process.

$$f'(pT, mT + lT) \triangleq f(mT + lT) \qquad \text{for } l \leq p$$
$$f'(pT, mT + lT) \triangleq 0 \qquad \text{for } l > p. \quad (55)$$

Using the unit step function,

$$f'(pT, mT + lT) = f(mT + lT)[1 - U(lT - pT)], \quad (56)$$

which is equivalent to (55). Eq. (54) may be written, noting the development above, to define the truncated, modified, z transform of $f(t)$.

$$F'^*(z, m, p) = \sum_{l=0}^{\infty} f'(pT, mT + lT) e^{-slT} \quad \text{for } z \triangleq e^{sT}. \quad (57)$$

The actual significance of (57) may be brought more clearly into focus by noting Barker's definition[6] for the "modified" z transform of a continuous time function. (Sometimes called the z, m transform). Extensive tables of z, m transforms exist in the literature.[6,10]

$$G_B^*(z, m) \triangleq z^{-1} \sum_{l=0}^{\infty} g(mT + lT) z^{-1} \quad \text{(after Barker)}. \quad (58)$$

Comparison of (58) with (57) shows that (57) is very similar to (58) except for a factor of z^{-1}. If tables of z, m transforms are used which are based on (58), a particular transform must be multiplied by z to be used in (59) below.

Since the variable filter in (57) results from the truncation process, similar to that used for $h_d'(t)$, we may express the right-hand member of (57) by taking the "improved" (a Barker transform, times z), modified, z transform of (56). The result is seen to involve complex convolution in z.

$$F'^*(z, m, p) = F^*(z, m)$$
$$- \frac{1}{2\pi j} \oint_{\Gamma_\zeta} F^*(y, m) \frac{(z/y)^{-p} dy}{(z/y - 1) y} \cdot \quad (59)$$

Γ_ζ is a contour enclosing the poles of $F^*(y, m)$, but excluding $y = z$. Note that $F^*(z, m)$ may simply be expressed in terms of the improved z, m transforms of the sampled-data channel dynamics. Since the digital filter responds only at sampling instants, its improved z, m transform is independent of m. Thus,

$$F^*(z, m) = D^*(z) G^*(z, m). \quad (60)$$

Noting the developments above, (53) may finally be written as

$$I_2 = -\frac{2}{2\pi j} \int_{-j\infty}^{j\infty} [\Phi_{vr}(s) H_d'(s, t) e^{-msT}] F'^*(e^{sT}, m, p) ds$$
$$= -2 \langle c_{d_i}(t) c_i(t) \rangle. \quad (61)$$

[10] E. I. Jury, "Additions to the modified z-transform method," 1957 WESCON Convention Record, pt. 4, pp. 136–156.

It is shown in Appendix II that (61) may be written

$$I_2 = -\frac{1}{\pi j}\int_I Z[\Phi_{vr}(s)H_d'(-s,t)e^{-smT}] \cdot F'^*(z,m,p)\frac{dz}{z}\cdot \tag{62}$$

The final step in deriving the ensemble mean-squared error as a function of system operating time is to consider (41c). The expectation $\langle r_i(nT)r_i(kT)\rangle$ is a stationary correlation function for n, $k\geq 0$, and a function of only the discrete time differences. Thus $\phi_{rr}(kT-nT)$ exists only for discrete arguments (n and k are positive integers). It may therefore be written as an inverse z transform:

$$\phi_{rr}(kT-nT) = \frac{1}{2\pi j}\int_I \Phi_{rr}^*(z)z^{k-n-1}dz. \tag{63}$$

Eq. (63) may be substituted into (41c) yielding, upon rearranging terms,

$$I_3 = \langle c_i^2(t)\rangle = \frac{1}{2\pi j}\int_I \Phi_{rr}^*(z)\sum_{n=0}^{n=p} f(t-nT)z^{-n} \cdot \sum_{k=0}^{k=p} f(t-kT)z^k\frac{dz}{z}\cdot \tag{64}$$

Following the development for $F'(z, m, p)$ in (53)–(60), we may write finally

$$\langle e_i^2(t)\rangle = I_1 + I_2 + I_3$$

$$= \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} \Phi_{vv}(s)H_d'(s,t)H_d'(-s,t)ds \tag{65a}$$

$$-\frac{1}{\pi j}\int_I Z[\Phi_{vr}(s)H_d'(-s,t)e^{-smT}]F'^*(z,m,p)\frac{d}{z} \tag{65b}$$

$$+\frac{1}{2\pi j}\int_I \Phi_{rr}^*(z)F'^*(z,m,p)F'^*(1/z,m,p)\frac{dz}{z}\cdot \tag{65c}$$

I is defined as the "semi-open" line segment, $|z|=1$ (the unit circle), $-\pi+A\leq\arg\, z<\pi+A$, taken in a counter-clockwise sense. A is an arbitrary real number.

Appendix II

Derivation of Inversion Integral for the Two-Sided z Transform

The latter two inversion integrals of (41) are of the form

$$I_1 = \frac{1}{2\pi j}\int_{-j\infty}^{j\infty} A(s)B^*(e^{sT})ds. \tag{66}$$

$A(s)$ and $B(e^{sT})$ are rational functions of s and e^{sT}, respectively, where the dependence on $t=pT+mT$ is neglected in this analysis since these variables are assumed to be constant for the evaluation of the integrals of (41). For practical reasons it has been tacitly assumed that the analytic strip for $A(s)B^*(e^{sT})$ includes the imaginary axis. Eq. (66) may be written as

$$\frac{1}{2\pi j}\sum_{n=-\infty}^{n=\infty}\int_{(n-1/2)j(2\pi/T)}^{(n+1/2)j(2\pi/T)} A(s)B^*(e^{sT})ds \tag{67}$$

or for

$$s\rightarrow s+jn\frac{2\pi}{T}$$

$$\frac{1}{2\pi j}\sum_{n=-\infty}^{n=\infty}\int_{-j(\pi/T)}^{j(\pi/T)} A\left(s+jn\frac{2\pi}{T}\right)B^*(e^{sT})ds \tag{68}$$

or

$$\frac{T}{2\pi j}\int_{-j(\pi/T)}^{j(\pi/T)} B^*(e^{sT})\left[\frac{1}{T}\sum_{n=-\infty}^{n=\infty} A\left(s+jn\frac{2\pi}{T}\right)\right]ds. \tag{69}$$

However, the bracketed term is recognized as the Poisson sum definition of a pulsed transform; thus (69) becomes

$$\frac{T}{2\pi j}\int_{-j(\pi/T)}^{j(\pi/T)} B^*(e^{sT})A^*(e^{sT})ds. \tag{70}$$

The transformation $z=e^{sT}$ in (70) results in

$$\frac{1}{2\pi j}\int_I B^*(z)A^*(z)\frac{dz}{z} \tag{71}$$

where I is the semi-open line segment defined as $|z|=1$, $-\pi\leq\arg\, z<\pi$ taken in a counter-clockwise sense. The semi-open definition for the arg z is necessary to preserve an analytic one-to-one mapping between s and z. The semi-open line segment may be closed either to the outside or the inside of the unit circle as shown in Fig. 9 dependent upon the convergence properties of the integrand. This may require splitting up the integrand as demonstrated in Section III by specific example.

It is readily seen that, in the event that the integrand $A^*(z)B^*(z)$ has a singular point along the negative real axis, either of these closures will result in an alteration of the value of the integral. In this event the original semi-open line segment may be chosen as $|z|=1$, $-\pi+A\leq\arg\, z<\pi+A$ where A real is chosen such that the closure may be made along a radial line along which the integrand is analytic.

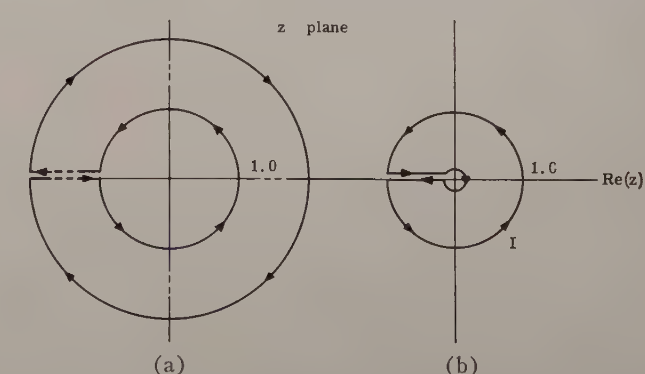

Fig. 9—Two closures of the semi-open line segment, I.

A Mathematical Representation of Hydraulic Servomechanisms*

J. J. RODDEN†

Summary—**This paper gives a basic representation of a high performance hydraulic servo actuator. The discussion gives a means of calculating the servo gain and rate limit characteristics as functions of the design parameters. Assumptions are discussed that would allow simplification of the equations for their application with computing equipment. Representation of the supply accumulator and pump is included. The effects of equipment nonlinearities can be incorporated into the basic model with appropriate modification of the computer program.**

The work of this paper has been applied to the design and analysis of missile flight control systems where the rate limit characteristic of the actuator is significant to dynamic performance.

Introduction

As the analysis of the Polaris flight control system proceeded, the mathematical models of the individual components were increasingly refined. The analysis continually considered more individual anomalies and nonlinearities as the system design became more fixed. During the early part of the preliminary analyses, the hydraulic servos were considered perfect (*i.e.*, controller position at all times equaled the autopilot position command). As the analysis continued, more complicated representations were used. The simplified dynamics of a first-order lag were simulated as the servo. Later, position limits were incorporated in the equations to simulate the mechanical stops in the servo actuator. Simple fixed limits on controller rate capability were then used to simulate the finite flow capability of the hydraulic pump-accumulator system.

The functional relationships of the parts of the servo system are shown in Fig. 1. From measurements of missile motion, the autopilot produces a command signal which is fed to the servo amplifier which in turn controls the servo valve flow. The servo valve flow positions the control surface which in turn produces a stabilizing force on the missile.

The following derivation is a further refinement of the servo description. It includes a variable rate limit as a function of the hydraulic fluid pressure, the nonlinear pressure-flow relations, the system friction, the pump, and the electronic circuit characteristics. This derivation gives a basic representation of the behavior of a hydraulic servo valve. Included for generality is a typical representation of a supply accumulator and pump. The effects of equipment nonlinearities, such as valve dead band or hysteresis and oil compressibility,

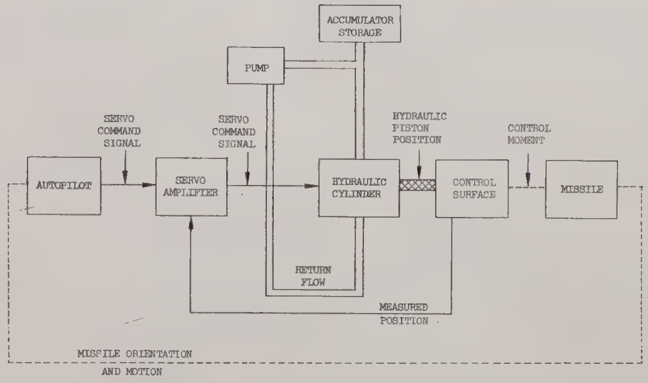

Fig. 1—Functional diagram of the hydraulic system in the flight control system.

are not discussed in this report; however, they can be incorporated into the basic equation with appropriate modification of the computer program.

Derivation of Servo Equations

A typical setup of a control valve regulating the pressure and flow to a driving piston is shown in Fig. 2.

The control valve is supplied with a high pressure source (P_A). The total flow (Q_T) is distributed between the piston flow (Q) and leakage flow (Q_L). A schematic representation of Fig. 2 is shown in Fig. 3.

For a positive valve displacement (ϵ), the flow is distributed as follows:

Valve	*Position*	*Flow*
A	open	piston flow plus leakage
B	closed to minimum	leakage flow
C	open	piston flow plus leakage
D	closed to minimum	leakage flow.

The significant pressure drops occur at the valve openings (orifices) within the valves *A*, *B*, *C*, and *D*. These pressure drops can be determined from the orifice flow equations.

The flow of a fluid through a valve opening (orifice) is presented in Fig. 4. Neglecting the dynamic $\frac{1}{2}\rho V^2$ terms, the flow of an incompressible fluid through a valve opening (orifice) is given by

$$Q_T = \sqrt{K} f(\epsilon) \sqrt{P_a - P_b}$$

where

$\sqrt{K}$ = a constant
ϵ = valve opening (orifice)
$f(\epsilon)$ = opening function
$(P_a - P_b)$ = pressure drop across orifice.

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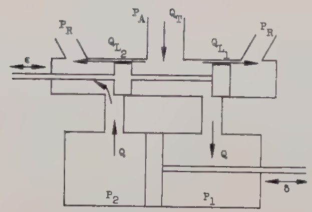


P_A = pressure source
P_R = pressure return
ϵ = valve displacement
Q_T = total flow
Q_L = leakage flow
Q = piston flow
P_1 = pressure on piston
P_2 = pressure on piston
δ = piston displacement

Fig. 2—Hydraulic control valve and actuator.

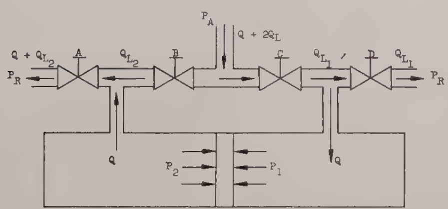
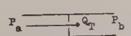


Fig. 3—Schematic representation of the hydraulic valve.



Q_T = flow through orifice
P_a = upstream pressure
P_b = downstream pressure

Fig. 4—Flow through an orifice.

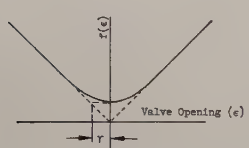


Fig. 5—Valve opening function.

The relationship between the opening function $[f(\epsilon)]$ and the valve opening (ϵ) is shown in Fig. 5.

The opening function $[f(\epsilon)]$ depends upon the valve opening (ϵ) and the minimum effective valve opening (γ); this latter value (γ) may be defined as the amount a perfect linear valve must be opened to yield the leakage flow (Q_L). The opening function $[f(\epsilon)]$ is approximated by

$$f(\epsilon) = \sqrt{\epsilon^2 + \gamma^2}.$$

Substituting this value in the orifice flow equation yields

$$Q_T = \sqrt{K}\sqrt{(P_A - P_R)(\epsilon^2 + \gamma^2)}.$$

Assuming a closed symmetrical control valve $(\epsilon = 0)$, the pressure drop across each of the closed valve openings $(A, B, C,$ and $D)$ is equal to $(P_A - P_R)/2$. The leakage flow (Q_L) is represented by

$$Q_L = \gamma\sqrt{K}\sqrt{(P_A - P_R)/2},$$

and the minimum effective valve opening (γ) is represented by

$$\gamma = \frac{Q_L}{\sqrt{K}\sqrt{(P_A - P_R)/2}}.$$

In the schematic diagram (Fig. 3), with the minimum valve openings effectively being γ, the pressure drops through the various valve openings are as follows:

Valve	*Position*	*Pressure Drop Through Valve*
A	open	$P_2 - P_R = \dfrac{(Q + Q_L)^2}{K(\epsilon^2 + \gamma^2)}$
B	closed	$P_A - P_2 = \dfrac{(Q_L)^2}{K\gamma^2}$
C	open	$P_A - P_1 = \dfrac{(Q + Q_L)^2}{K(\epsilon^2 + \gamma^2)}$
D	closed	$P_1 - P_R = \dfrac{(Q_L)^2}{K\gamma^2}$

The pressure drop from the supply pressure to the return pressure $(P_A - P_R)$ equals the sum of the pressure drops across valve C $(P_A - P_1)$, across the piston $(P_1 - P_2)$, and across valve A $(P_2 - P_R)$. From this relationship, the equation for the driving force on the piston $(P_1 - P_2)$ can be derived as follows:

$$\begin{aligned} P_1 - P_2 &= (P_A - P_R) - (P_2 - P_R) - (P_A - P_1) \\ &= (P_A - P_R) - 2\left[\frac{(Q + Q_L)^2}{K(\epsilon^2 + \gamma^2)}\right]. \end{aligned}$$

The differential pressures for a positive value displacement are shown in Fig. 6.

Opening valves B and D and closing valves A and C results in a negative displacement in the control valve position and an interchange in the ram pressures P_1 and P_2. The differential pressures for both positive and negative displacements of the control valve are shown in Fig. 7. The differential driving pressure can be written as follows:

$$P_1 - P_2 = \left[(P_A - P_R) - 2\frac{(Q + Q_L)^2}{K(\epsilon^2 + \gamma^2)}\right] \operatorname{sgn}(\epsilon)$$

where the function sgn (ϵ) is $(+1)$ or (-1) corresponding to the sign of (ϵ).

It is known that the valve leakage is small and that variations in its value have little influence on the performance of the valve. For the remainder of this analysis the valve leakage (Q_L) will be assumed to be equal to the leakage for a closed valve as shown by

$$Q_L = \gamma\sqrt{K}\sqrt{(P_A - P_R)_{\max}/2}.$$

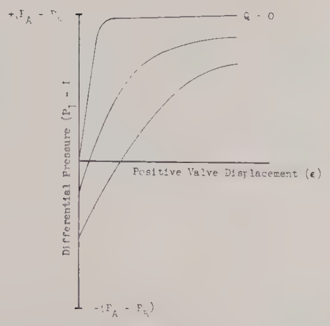


Fig. 6—Differential pressure for positive opening.

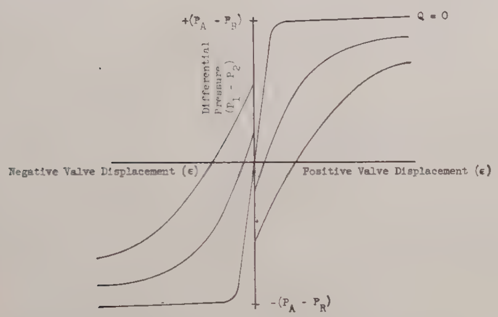


Fig. 7—Differential pressure vs valve position.

The corresponding minimum valve opening (γ) is represented by

$$\gamma = \frac{Q_L}{\sqrt{K}\sqrt{(P_A - P_R)_{\max}/2}}.$$

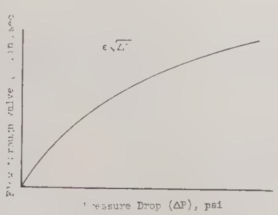
Substituting this relationship in the differential pressure equation and neglecting the product of the valve leakage yields

$$(P_1 - P_2) = \left[\frac{(P_A - P_R)K\epsilon^2 - 2Q^2}{K\left(\epsilon^2 + \dfrac{2Q_L^2}{K(P_A - P_R)_{\max}}\right)}\right] \operatorname{sgn}(\epsilon).$$

Measurements of the control valve characteristic are made under the following conditions: 1) The piston cylinder is by-passed ($P_1 = P_2$), 2) the pressure drop across the servo valve ($P_A - P_R$) is varied, and 3) flow measurements are made with the control valve wide open. The relationship between the differential pressure ($P_A - P_R$) and the flow (Q) through the control valve is shown in Fig. 8. The piston flow rate (Q) for a variable valve opening is equal to $c\epsilon\sqrt{\Delta P}$; the piston flow rate (Q) for a wide open valve is equal to $c\sqrt{\Delta P}$.

The relationship between the valve characteristic (c) and the value of K (previously used in this paper) is shown by

$$c = \sqrt{K/2}.$$



Q = flow through valve
c = valve characteristic
ΔP = pressure drop across control valve
ϵ = fraction of valve opening $= \delta_\epsilon/\delta\epsilon_{\max}$ and is equal to 1.0 for a wide open valve

Fig. 8—Flow characteristic through valve.

The relationship between the fraction of valve opening (ϵ) and the piston displacement (δ) is shown by

$$\epsilon = \delta_\epsilon/\delta_{\epsilon_{\max}}.$$

Substituting these relationships in the differential pressure equation yields

$$(P_1 - P_2) = \left[\frac{(P_A - P_R)\delta_\epsilon^2 - \left(\dfrac{Q\delta_{\epsilon_{\max}}}{c}\right)^2}{\delta_\epsilon^2 + \dfrac{\left(\dfrac{\delta_{\epsilon_{\max}}Q_L}{c}\right)^2}{(P_A - P_R)_{\max}}}\right] \operatorname{sgn}(\delta_\epsilon)$$

In the case where the motion of the piston is used to deflect the control surface, a direct relationship exists between the piston displacement and the deflection of the control surface (δ); similarly, a direct relationship exists between the piston flow (Q) and the rate of deflection of the control surface ($\dot{\delta}$). This latter relationship is expressed by

$$Q = b\,|\,\dot{\delta}\,|\,.$$

The moment on the control surface is found by multiplying the inertia of the control surface (J) by the angular acceleration ($\ddot{\delta}$) and equating this product to the difference between the torque due to the differential pressure ($P_1 - P_2$) and the friction load (T), as shown by

$$J\ddot{\delta} = A_R a(P_1 - P_2) - (T)\operatorname{sgn}(\dot{\delta})$$

where

J = inertia of the control surface
$\ddot{\delta}$ = acceleration of the control surface
a = lever arm around control surface pivot
A_R = area of piston
T = friction load (coulomb).

The friction load is assumed to be of the sliding or coulomb type, and is constant in magnitude and opposed to the motion of the piston.

The acceleration of the control surface is obtained by substituting the differential pressure previously derived into the above equation and dividing by the inertia of the control surface, as shown by

$$\ddot{\delta} = \frac{A_R a}{J}(P_A - P_R)\left[\frac{\delta_\epsilon^2 - \left(\frac{b\delta_{\epsilon_{\max}}}{c}\right)^2 \frac{\dot{\delta}^2}{(P_A - P_R)}}{\delta_\epsilon^2 + \frac{\left(\frac{\delta_{\epsilon_{\max}} Q_L}{c}\right)^2}{(P_A - P_R)_{\max}}}\right] \operatorname{sgn}(\delta_\epsilon) - \frac{T}{J}\operatorname{sgn}(\dot{\delta}).$$

High-Pressure Source and Pump

Nominally some kind of high-pressure storage is used to supply fluid to the servo control valve. This may exist as an accumulator tank with a pump supply as shown in Fig. 9.

The accumulator is initially precharged with air under pressure $P_{\min}$ and then partially filled with hydraulic oil which further compresses the air. The accumulator pressure is the pressure of the stored fluid and gas; it can be found from gas-law considerations, as shown by

$$P_A(V_{\text{air}})^K = C_a$$

where

$V_{\text{total}} = V_{\text{air}} + V_{\text{oil}}$
K = gas constant = 1.4 for air in an adiabatic expansion
P_A = pressure in accumulator
C_a = constant for the system $= P_{\min}(V_{\text{total}})^K$;

thus

$$P_A = \frac{C_a}{(V_{\text{total}} - V_{\text{oil}})^K}.$$

For the high-pressure systems used with servo actuators, the relationship between P_A and V_{oil} can be linearized as shown by

$$P_A = \left(\frac{P_s - P_m}{V_{\text{oil}_{\max}}}\right) V_{\text{oil}} + P_{\min} \qquad \text{for } P_A > P_{\min}.$$

The flow from the pump is in general dependent on the system pressure. Fig. 10 shows a typical characteristic.

$$Q_P = \begin{cases} C_{q_1}(P_S - P_A) & \text{for } C_{q_1}(P_S - P_A) \le Q_{P_1} \\ Q_{P_1} + C_{q_2}(P_B - P_A) & \text{for } C_{q_1}(P_S - P_A) > Q_{P_1}. \end{cases}$$

Pump Without Accumulator

For those flight control systems without an accumulator for storage of hydraulic fluid, the flow rate of fluid in the control cylinders is exactly equal to the pump output. The pressure of that fluid is found directly from the pump characteristic as shown in Fig. 10. In the previous case, the pump output in cubic inches of fluid per second was found as a function of the line pressure. In this case with no accumulator storage, the line pressure is found as a function of the total flow rate as shown in Fig. 11. Whether the line pressure is to be determined as a function of the accumulator volume or the system flow depends on the value of the line pressure. When the pressure is greater than the accumulator precharge pressure, the system is operating off the accumulator. When the accumulator is bottomed (drained) and the line pressure falls below the accumulator precharge, the flow comes directly from the pump. Whenever the pressure rises above the precharge level, operation resumes from the accumulator.

$$P_A = \begin{cases} P_S - \dfrac{Q_P}{C_{q_1}} & \text{for } Q_P < Q_{P_1} \\ P_B - \dfrac{(Q_P - Q_{P_1})}{C_{q_2}} & \text{for } Q_P > Q_{P_1}. \end{cases}$$

The pump equations can take on other forms, but these can be considered typical.

Typical ranges of values for a high performance servo are given in Appendix I. A summary of the servo, pump, and accumulator equations, along with a schematic

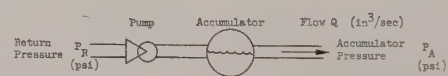


Fig. 9—High-pressure system.

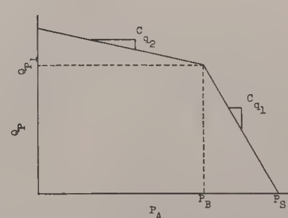


Fig. 10—Pump flow characteristic.

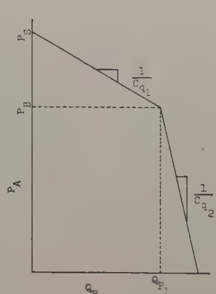


Fig. 11—Inverted pump characteristic.

computer diagram, is presented in Appendix II. When these equations are simulated on an analog computer, the resulting data can be plotted to yield a frequency response curve. The frequency response curve for a typical servo design with coulomb friction is shown in Fig. 12.

Servo Gain and Rate Limit

The servo nominally performs as an integrating device. This means that a unit signal into the servo will produce a rate in the output proportional to the input signal. The relationship between the position of the control surface and the input signal is shown schematically in Fig. 13.

This relationship is expressed by

$$\dot{\delta} = K_v \delta_\epsilon.$$

In a hydraulic system, this relationship is met when the acceleration of the output is zero, *i.e.*, coast rate dependent on input alone. Neglecting viscous damping and return pressure, the acceleration for a positive δ_ϵ can be expressed by

$$\ddot{\delta} = \frac{A_R a}{J} P_A \left[\frac{\delta_\epsilon{}^2 - \left(\frac{b\delta_{\epsilon_{\max}}}{c} \right)^2 \frac{\dot{\delta}_\epsilon{}^2}{P_A}}{\delta_\epsilon{}^2 + \gamma^2} \right] - \frac{T}{J}.$$

When the output acceleration is zero, the output rate is proportional to the signal input up to the rate limit. The output rate is expressed by

$$\dot{\delta} = \left\{ \frac{\sqrt{P_A}}{\left(\frac{b\delta_{\epsilon_{\max}}}{c} \right)} \sqrt{1 - \frac{T}{A_R a P_A}} \right\} \delta_\epsilon$$

where

$\dot{\delta}$ = output rate
δ_ϵ = signal input.

The servo gain can be expressed by

$$K_v = \frac{\sqrt{P_A}}{\left(\frac{b\delta_{\epsilon_{\max}}}{c} \right)} \sqrt{1 - \frac{T}{A_R a P_A}} \; \frac{\text{deg/sec}}{\text{deg}}.$$

As the maximum rate corresponds to a wide-open valve ($\delta_\epsilon = \delta_{\epsilon_{\max}}$), the relation between the servo gain and the rate limit can be expressed by

$$K_V = \frac{\dot{\delta}_L}{\delta_{\epsilon_{\max}}}$$

where

K_V = gain in deg/second/deg
$\dot{\delta}_L$ = rate limit
$\delta_{\epsilon_{\max}}$ = maximum valve input.

Fig. 14 shows a typical rate limit characteristic as a function of system pressure for a high performance servo actuator. Laboratory tests on flight hardware have shown close correlation between predicted and measured servo performance.

Appendix III summarizes equations of the modified servo dynamics.

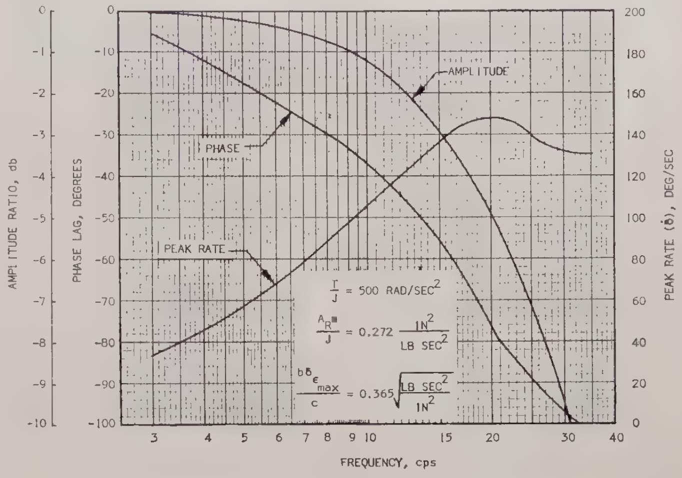
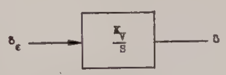


Fig. 12—Frequency response of a hydraulic system (one servo with 2° amplitude input).



δ = output (control surface deflection)
$\dot{\delta}$ = output rate
δ_ϵ = input signal to servo
K_v = servo gain
s = complex variable

Fig. 13—Simplified valve representation.

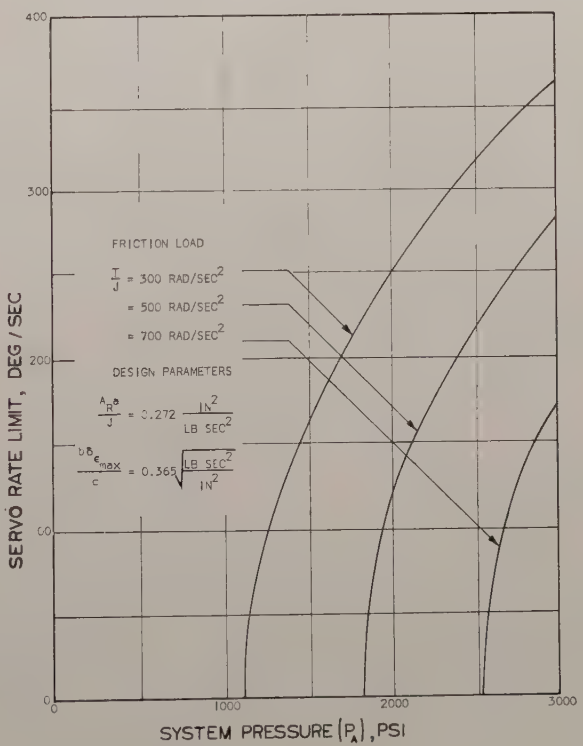


Fig. 14—Servo rate limit vs system pressure as a function of friction load.

Conclusion

The simplified hydraulic servo equations are readily applicable in analog and digital trajectory programs. The simulation can then be used to investigate missile maneuvers in which control surface rate limit characteristics are significant to performance, such as missile launch and wind shear transients. A significant advantage of using the servo model is to allow the analyst to determine directly the effect of hydraulic subsystem parameters on the flight control system operation.

Appendix I

Glossary of Terms and Typical Values

Symbol	*Description*	*Typical Values*
δ	control surface position	± 30 deg
$\dot{\delta}$	control surface rate	± 500 deg/second
$\ddot{\delta}$	control surface acceleration	$\pm 50{,}000$ deg/second^2
δ_ϵ	control surface error signal	± 5 deg
δ_c	control surface command	± 30 deg
P_A	accumulator pressure	2000–3000 psi
P_S	supply pressure	3000 psi
P_m	precharge pressure	500–1500 psi
P_R	reservoir pressure	0–100 psi
V_T	total volume in accumulator	10–20 in^3
V_{air}	air volume in accumulator	10–20 in^3
V_{oil}	oil volume in accumulator	0–5 in^3
Q_L	valve leakage	0.01–0.05 in^3/second
Q_p	pump flow	4–8 in^3/second
b	flow per unit stroke	0.02–0.08 in^3/deg
K_v	open-loop servo gain	0–100 deg/second/deg
ϵ	generalized valve opening	——
γ	minimum effective valve closure	——

Appendix II

Equations of the Servo Hydraulic System

Neglecting the return pressure (P_R), the servo equations are as follows:

$$\ddot{\delta}_\Delta = + \frac{A_R a}{J}(P_1 - P_2)\operatorname{sgn}(\delta_\epsilon) - \frac{T}{J}\operatorname{sgn}(\dot{\delta})$$

$$(P_1 - P_2) = P_A\left[\frac{\delta_\epsilon^2 - \left(\dfrac{b\delta_{\epsilon_{\max}}}{c}\right)^2 \dfrac{\dot{\delta}^2}{P_A}}{(\delta_\epsilon^2 + \gamma^2)}\right]$$

$$Q = b\,|\,\dot{\delta}\,|$$

$$\delta_\epsilon = \begin{cases} +\delta_{\epsilon_{\max}} & \text{for } (\delta_c - \delta) > \delta_{\epsilon_{\max}} \\ (\delta_c - \delta) & \text{for } |\,\delta_c - \delta\,| < \delta_{\epsilon_{\max}} \\ -\delta_{\epsilon_{\max}} & \text{for } (\delta_c - \delta) < -\delta_{\epsilon_{\max}}. \end{cases}$$

Position Limits

$$\dot{\delta} = \begin{cases} 0 & \text{for } |\,\delta\,| > \delta_{\max} \\ \dot{\delta}_\Delta & \text{for } |\,\delta\,| < \delta_{\max}. \end{cases}$$

Pump Equations

For variable displacement pump:

$$Q_P = \begin{cases} C_{q_1}(P_S - P_A) & \text{for } C_{q_1}(P_S - P_A) < Q_{P_1} \\ Q_{P_1} + C_{q_1}(P_B - P_A) & \text{for } C_{q_1}(P_S - P_A) > Q_{P_1}. \end{cases}$$

For constant displacement pump:

$$Q_P = \begin{cases} C_P & \text{for } P_A < P_S \\ 0 & \text{for } P_A > P_S. \end{cases}$$

For pump without accumulator:

$$P_A = \begin{cases} P_S - \dfrac{Q_P}{C_{q_1}} & \text{for } Q_P < Q_{P_1} \\ P_B - \dfrac{(Q_P - Q_{P_1})}{C_{q_1}} & \text{for } Q_P > Q_{P_1}. \end{cases}$$

Accumulator Equations

$$\dot{V}_{\text{oil}} = Q_P - \sum_{i=1}^{n} Q_i$$

where n is the number of servos operating.

$$V_{\text{air}} + V_{\text{oil}} = V_{\text{total}}$$

$$P_A = \frac{C_a}{(V_{\text{air}})^k}.$$

The last equation can be linearized to:

$$P_A = \left(\frac{P_s - P_{\min}}{V_{\text{oil}_{\max}}}\right)V_{\text{oil}} + P_{\min} \qquad \text{for } P_A > P_{\min}.$$

These equations are shown schematically in Fig. 15.

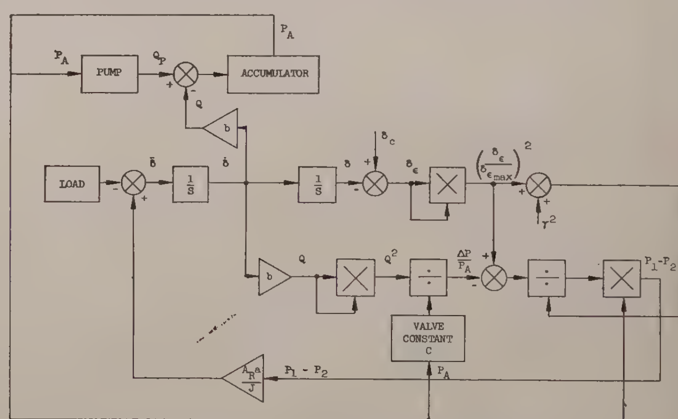


Fig. 15—Schematic diagram for servo equations.

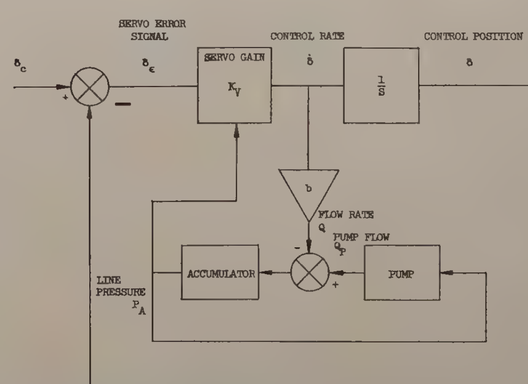


Fig. 16—Simplified servo dynamics.

APPENDIX III

MODIFIED SERVO DYNAMICS

$$\dot{\delta} = K_V \delta_\epsilon$$

$$\dot{\delta} = \begin{cases} 0 & \text{for } |\delta| > \delta_{\max} \\ \delta_\Delta & \text{for } |\delta| < \delta_{\max} \end{cases}$$

$$\delta_\epsilon = \begin{cases} +\delta_{\epsilon_{\max}} & \text{for } (\delta_c - \delta) > \delta_{\epsilon_{\max}} \\ (\delta_c - \delta) & \text{for } |\delta_c - \delta| < \delta_{\epsilon_{\max}} \\ -\delta_{\epsilon_{\max}} & \text{for } (\delta_c - \delta) < -\delta_{\epsilon_{\max}} \end{cases}$$

$$K_V = f(P_A) = \frac{\sqrt{P_A}}{\left(\frac{\delta b_{\epsilon_{\max}}}{c}\right)} \sqrt{1 - \frac{T}{A_R a P_A}}$$

The pump and accumulator equations are presented in Appendix II.

The simplified servo dynamics are shown schematically in Fig. 16.

ACKNOWLEDGMENT

The author wishes to thank B. C. Axley for his helpfulness in the development of this paper.

A General Method for Deriving the Describing Functions for a Certain Class of Nonlinearities*

RANGASAMI SRIDHAR†

Summary—Since Goldfarb's[1] original work on describing functions, a considerable number of papers have been published in which the describing functions of particular nonlinearities have been derived. It appears however that little effort has been made to classify the nonlinearities. Since the describing function method is one of the more powerful methods available at present to analyze nonlinear feedback systems, it appears desirable to collect the expressions for the describing functions of a few different types of nonlinearities in one paper.

It is the purpose of this paper to derive the describing functions of two general types of nonlinearities and show how the describing functions of many other practical types of nonlinearities for which the describing function analysis is valid naturally follow.

DERIVATION OF THE DESCRIBING FUNCTIONS

THE two general types of nonlinearities to be considered here will be designated as the piecewise linear type nonlinearity and the polynomial type nonlinearity. The input-output characteristic of the piecewise linear type nonlinearity consists of straight line segments, and the input-output characteristic of the polynomial type nonlinearity is described by a polynomial. It is assumed that the input-output characteristics of all the nonlinearities considered here are odd functions with respect to the input. The describing functions of the two general types of nonlinearities will be separately derived next.

Describing Function of the General Piecewise Linear Type Nonlinearity

The type of nonlinearity considered (Fig. 1), for the derivation of the describing function probably does not occur as the characteristic of a single nonlinearity in practice. However, it is quite conceivable that Fig. 1 might represent the combined characteristic of multiple nonlinearities in cascade. The chief reason for choosing this nonlinearity is that many other piecewise linear type nonlinearities to which the describing function analysis is applicable are modifications of this general type, as will be shown below. In further discussions, the general type of nonlinearity shown in Fig. 1 will be referred to as the type A nonlinearity.

Let the input to the nonlinear element be x and the output by y. Then y can be represented as a function of x; *i.e.*, $y = f(x)$ where $f(x)$ is the input-output characteristic shown in Fig. 1.

The describing function or equivalent admittance of a nonlinear element is defined as the complex ratio of the amplitude of the fundamental of the output to the amplitude of the input when the input is sinusoidal.

Using the same notation as Goldfarb, the describing function $J_{n1}(A)$ is

$$J_{n1}(A) = g_e(A) + jb_e(A) \tag{1}$$

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[1] L. C. Goldfarb, "On some nonlinear phenomena in regulatory systems," in "Frequency Response," R. Oldenburger, Ed., The Macmillan Co., New York, N. Y., pp. 239–259; 1956. This is an English translation of the original article which appeared in the Russian journal *Avtomatika i Telemekhanika*, vol. 8, pp. 349–383; September–October, 1947.

where

$$g_e(A) = \frac{1}{\pi A}\int_0^{2\pi} f(A \sin\theta)\cdot\sin\theta\cdot d\theta \qquad (2)$$

$$b_e(A) = \frac{1}{\pi A}\int_0^{2\pi} f(A \sin\theta)\cdot\cos\theta\cdot d\theta. \qquad (3)$$

Eq. (1) may also be written in the following simple form:

$$J_{n1}(A) = \frac{j}{\pi A}\int_0^{2\pi} f(A \sin\theta)\cdot\exp(-j\theta)\cdot d\theta. \qquad (4)$$

In this definition it is assumed that the input x to the nonlinearity is $A \sin\theta$, where $\theta = \omega t$. Here ω is the frequency of the sine wave and t is time.

Referring to the output waveform shown in Fig. 1, it is seen that the integral in the right side of (4) can be broken up into a sum of twelve integrals since the expression for the output consists of twelve different equations [(18) to (29) in Appendix I], each valid over only a certain interval.

Thus, expressing $f(A \sin\theta)$ appropriately using (18) to (29), (2) and (3) can be rewritten as shown in Appendix II to yield (5) and (6).

$$\mathrm{Re}\{J_{n1}(A)\} = g_e(A)$$

$$= \frac{1}{\pi}\Big[-m_1(\theta_1+\theta_4) + 2(m_1-m_3)\cdot\theta_2 + (m_1-m_2)$$

$$\cdot(\theta_3-\theta_5) + \pi m_3 + \frac{m_1}{2}(\sin 2\theta_1 + \sin 2\theta_4) - (m_1-m_3)$$

$$\cdot\sin 2\theta_2 - \frac{(m_1-m_2)}{2}\cdot(\sin 2\theta_3 - \sin 2\theta_5) - 2m_1\frac{a}{A}\cos\theta_1$$

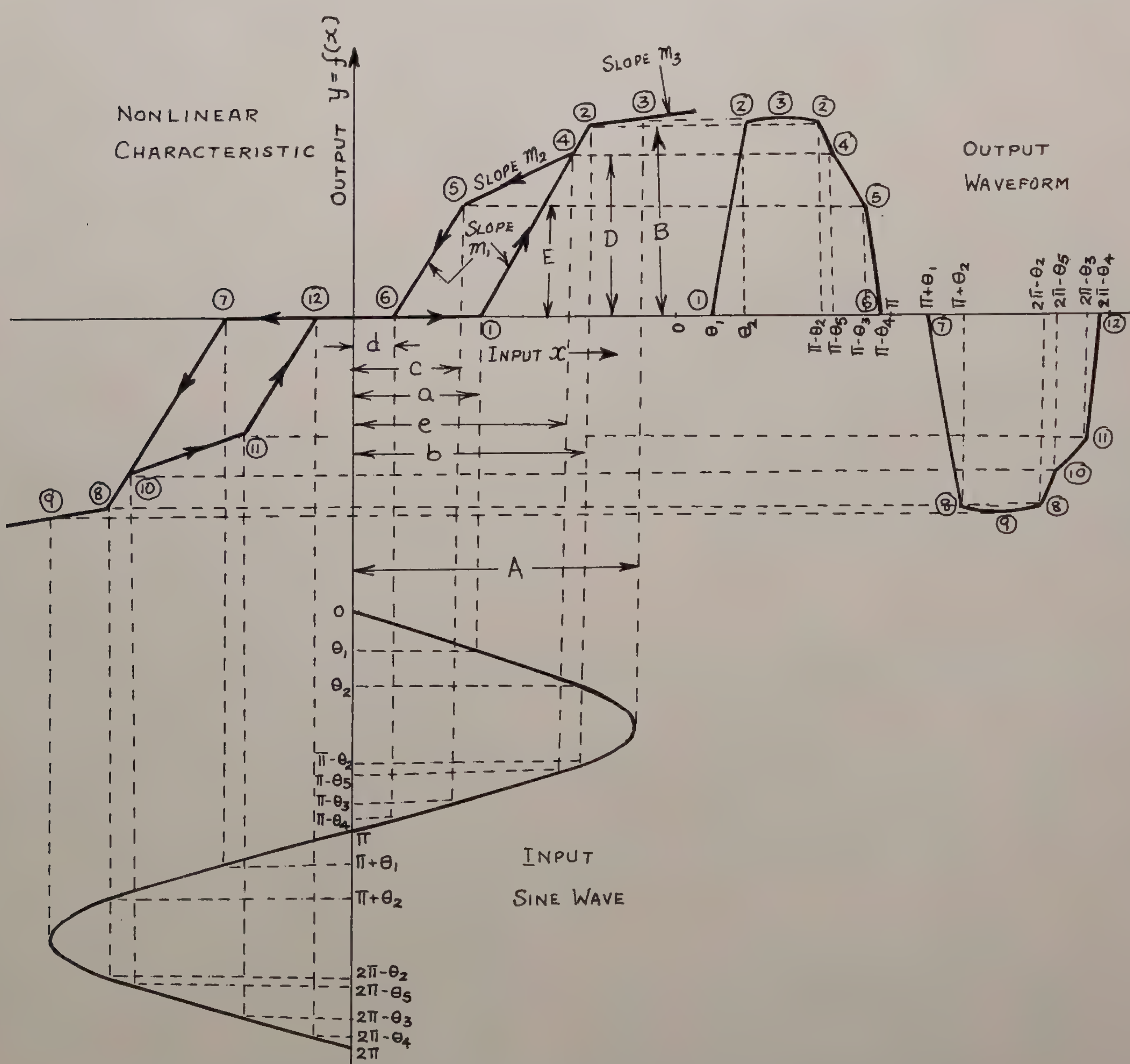


Fig. 1—Response of the general piecewise linear nonlinearity (type A) to a sine wave.

$$+ 4(m_1 - m_3)\frac{b}{A}\cos\theta_2 + 2(m_1 - m_2)\frac{c}{A}\cos\theta_3$$

$$\left. - 2m_1\frac{d}{A}\cos\theta_4 - 2(m_1 - m_2)\frac{e}{A}\cos\theta_5\right] \tag{5}$$

$$\mathrm{Im}\{J_{n1}(A)\} = b_e(A)$$

$$= \frac{1}{\pi}\left[\frac{m_1}{2}(\cos 2\theta_1 - \cos 2\theta_4) + \frac{(m_1 - m_2)}{2}\right.$$

$$\cdot(\cos 2\theta_3 - \cos 2\theta_5) + 2m_1\frac{a}{A}\sin\theta_1$$

$$+ 2(m_1 - m_2)\frac{c}{A}\sin\theta_3 - 2m_1\frac{d}{A}\sin\theta_4$$

$$\left. - 2(m_1 - m_2)\frac{e}{A}\sin\theta_5\right]. \tag{6}$$

Notice that θ_1, θ_2, θ_3, θ_4, and θ_5 have been defined so that they all lie in the first quadrant.

Thus, the describing function for the type A nonlinearity is given by (5) and (6). However, it is sometimes convenient to express the describing function in the polar form. Eqs. (5) and (6) may be rewritten in the following form:

$$J_{n1}(A) = |J_{n1}(A)| \cdot \exp\{j \cdot r(A)\} \tag{7}$$

where

$$J_{n1}(A) = \sqrt{g_e^2(A) + b_e^2(A)} \tag{8}$$

and

$$r(A) = \arctan\frac{b_e(A)}{g_e(A)}. \tag{9}$$

From the general equations (5) and (6), the describing functions of a large variety of piecewise linear nonlinearities can be derived by considering them as modifications of the type A nonlinearity.

Modifications of the Type A Nonlinearity: While considering the modifications of the type A nonlinearity, the nonlinearities have been divided into two categories, *viz.*, nonlinearities without memory and nonlinearities with memory.[2] The various modifications are shown in Tables I and II which include the describing functions for nine nonlinearities without memory and fourteen nonlinearities with memory.

[2] Nonlinearities without memory are defined as those whose input-output characteristics are single-valued, and nonlinearities with memory are defined as those whose input-output characteristics are multivalued.

TABLE I

NONLINEARITIES WITH MEMORY

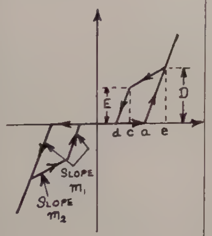
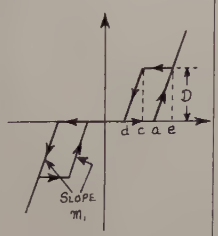
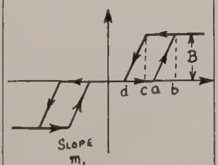
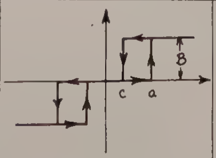
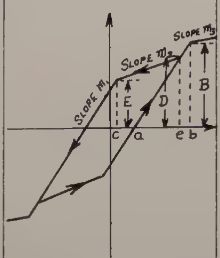
Alphabetical Classification	Characteristic	Modification of	Parameters Modified	Describing Function
B		A	$m_3 = m_1$	$g_e(A) = \frac{1}{\pi}\left[\pi m_1 - m_1(\theta_1 + \theta_4) + (m_1 - m_2)\cdot(\theta_3 - \theta_5) + \frac{m_1}{2}(\sin 2\theta_1 + \sin 2\theta_4) - \frac{(m_1 - m_2)}{2}(\sin 2\theta_3 - \sin 2\theta_5) - 2m_1\frac{a}{A}\cos\theta_1 + 2(m_1 - m_2)\frac{c}{A}\cos\theta_3 - 2m_1\frac{d}{A}\cos\theta_4 - 2(m_1 - m_2)\frac{e}{A}\cos\theta_5\right]$ $b_e(A) = \frac{1}{\pi}\left[\frac{m_1}{2}(\cos 2\theta_1 - \cos 2\theta_4) + \frac{(m_1 - m_2)}{2}(\cos 2\theta_3 - \cos 2\theta_5) + 2m_1\frac{a}{A}\sin\theta_1 + 2(m_1 - m_2)\frac{c}{A}\sin\theta_3 - 2m_1\frac{d}{A}\sin\theta_4 - 2(m_1 - m_2)\frac{e}{A}\sin\theta_5\right]$
C		B	$m_2 = 0$	$g_e(A) = \frac{m_1}{\pi}\left[\pi - (\theta_1 + \theta_4) + (\theta_3 - \theta_5) + \frac{1}{2}(\sin 2\theta_1 - \sin 2\theta_3 + \sin 2\theta_4 + \sin 2\theta_5) - 2\left(\frac{a}{A}\cos\theta_1 - \frac{c}{A}\cos\theta_3 + \frac{d}{A}\cos\theta_4 + \frac{e}{A}\cos\theta_5\right]$ $b_e(A) = \frac{m_1}{\pi}\left[\frac{1}{2}(\cos 2\theta_1 + \cos 2\theta_3 - \cos 2\theta_4 - \cos 2\theta_5) + 2\left(\frac{a}{A}\sin\theta_1 + \frac{c}{A}\sin\theta_3 - \frac{d}{A}\sin\theta_4 - \frac{e}{A}\sin\theta_5\right)\right]$
F		E	$e = b$	$g_e(A) = \frac{m_1}{\pi}\left[(-\theta_1 + \theta_2 + \theta_3 - \theta_4) + \frac{1}{2}(\sin 2\theta_1 - \sin 2\theta_2 - \sin 2\theta_3 + \sin 2\theta_4) + \frac{2}{A}(-a\cos\theta_1 + b\cos\theta_2 + c\cos\theta_3 - d\cos\theta_4)\right]$ $b_e(A) = \frac{m_1}{\pi}\left[\frac{1}{2}(\cos 2\theta_1 - \cos 2\theta_2 + \cos 2\theta_3 - \cos 2\theta_4) + \frac{2}{A}(a\sin\theta_1 - b\sin\theta_2 + c\sin\theta_3 - d\sin\theta_4)\right]$
G		F	$b = a$ $d = c$	$g_e(A) = \frac{2B}{\pi A}(\cos\theta_1 + \cos\theta_2)$ $b_e(A) = -\frac{2B}{\pi A}(\sin\theta_1 - \sin\theta_3)$
H		A	$d = -a$	$g_e(A) = \frac{1}{\pi}\left[2(m_1 - m_3)\theta_2 + (m_1 - m_2)\cdot(\theta_3 - \theta_5) + \pi m_3 - (m_1 - m_3)\sin 2\theta_2 - \frac{(m_1 - m_2)}{2}(\sin 2\theta_3 - \sin 2\theta_5) + 4(m_1 - m_3)\cdot\frac{b}{A}\cos\theta_2 + 2(m_1 - m_2)\frac{c}{A}\cos\theta_3 - 2(m_1 - m_2)\frac{e}{A}\cos\theta_5\right]$ $b_e(A) = \frac{1}{\pi}\left[\frac{(m_1 - m_2)}{2}(\cos 2\theta_3 - \cos 2\theta_5) + 2(m_1 - m_2)\frac{c}{A}\sin\theta_3 - 2(m_1 - m_2)\frac{e}{A}\sin\theta_5\right]$

TABLE 1 (*Cont'd*)

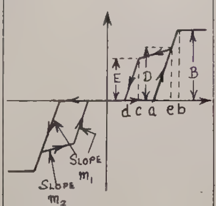
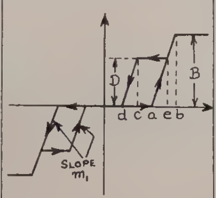
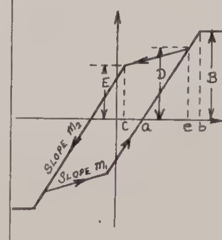
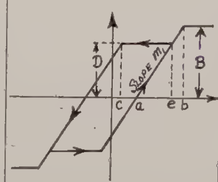
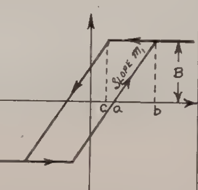
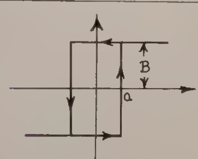
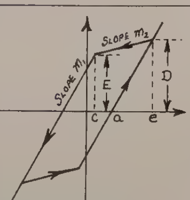
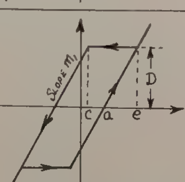
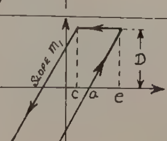
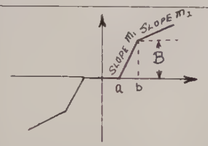
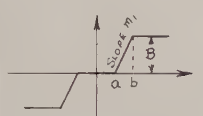
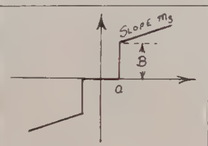
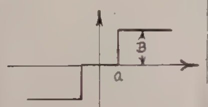
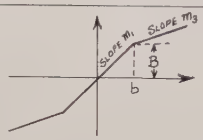
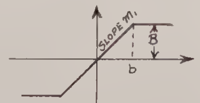
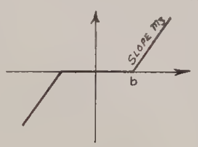
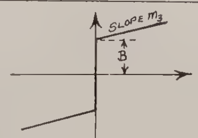
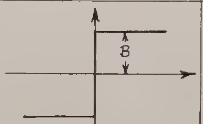
Alphabetical Classification	Characteristic	Modification of	Parameters Modified	Describing Function
D		A	$m_3=0$	$g_e(A) = \frac{1}{\pi}\Big[m_1(-\theta_1+2\theta_2-\theta_4)+(m_1-m_2)\cdot(\theta_3-\theta_5)+\frac{m_1}{2}(\sin 2\theta_1 - 2\sin 2\theta_2 + \sin 2\theta_4) - \frac{(m_1-m_2)}{2}(\sin 2\theta_3 - \sin 2\theta_5) - 2m_1\frac{a}{A}\cdot\cos\theta_1 + 4m_1\frac{b}{A}\cos\theta_2 + 2(m_1-m_2)\frac{c}{A}\cos\theta_3 - 2m_1\frac{d}{A}\cos\theta_4 - 2(m_1-m_2)\frac{e}{A}\cos\theta_5\Big]$ $b_e(A) = \frac{1}{\pi}\Big[\frac{m_1}{2}(\cos 2\theta_1 - \cos 2\theta_4) + \frac{(m_1-m_2)}{2}(\cos 2\theta_3 - \cos 2\theta_5) + 2m_1\frac{a}{A}\sin\theta_1 + 2(m_1-m_2)\frac{c}{A}\sin\theta_3 - 2m_1\frac{d}{A}\sin\theta_4 - 2(m_1-m_2)\frac{e}{A}\sin\theta_5\Big]$
E		D	$m_2=0$	$g_e(A) = \frac{m_1}{\pi}\Big[(-\theta_1+2\theta_2+\theta_3-\theta_4-\theta_5)+\frac{1}{2}(\sin 2\theta_1 - 2\sin 2\theta_2 - \sin 2\theta_3 + \sin 2\theta_4 + \sin 2\theta_5) + \frac{2}{A}(-a\cos\theta_1 + 2b\cos\theta_2 + c\cos\theta_3 - d\cos\theta_4 - e\cos\theta_5)\Big]$ $b_e(A) = \frac{m_1}{\pi}\Big[\frac{1}{2}(\cos 2\theta_1 + \cos 2\theta_3 - \cos 2\theta_4 - \cos 2\theta_5) + \frac{2}{A}(a\sin\theta_1 + c\sin\theta_3 - d\sin\theta_4 - e\sin\theta_5)\Big]$
I		H	$m_3=0$	$g_e(A) = \frac{1}{\pi}\Big[2m_1\theta_2+(m_1-m_2)(\theta_3-\theta_5) - m_1\sin 2\theta_2 - \frac{(m_1-m_2)}{2}(\sin 2\theta_5 - \sin 2\theta_5) + 4m_1\frac{b}{A}\cos\theta_2 + 2(m_1-m_2)\frac{c}{A}\cos\theta_3 - 2(m_1-m_2)\frac{e}{A}\cos\theta_5\Big]$ $b_e(A) = \frac{1}{\pi}(m_1-m_2)\Big[\frac{1}{2}(\cos 2\theta_3 - \cos 2\theta_5) + 2\frac{c}{A}\sin\theta_3 - 2\frac{e}{A}\sin\theta_5\Big]$
J		I	$m_2=0$	$g_e(A) = \frac{m_1}{\pi}\Big[2\theta_2+\theta_3-\theta_5-\sin 2\theta_2 - \frac{1}{2}(\sin 2\theta_3 - \sin 2\theta_5) + \frac{2}{A}(2b\cos\theta_2 + c\cos\theta_3 - e\cos\theta_5)\Big]$ $b_e(A) = \frac{m_1}{\pi}\Big[\frac{1}{2}(\cos 2\theta_3 - \cos 2\theta_5) + \frac{2}{A}(c\sin\theta_3 - e\sin\theta_5)\Big]$
K		J	$e=b$	$g_e(A) = \frac{m_1}{\pi}\Big[(\theta_2+\theta_3) - \frac{1}{2}(\sin 2\theta_2 + \sin 2\theta_3) + \frac{2}{A}(b\cos\theta_2 + c\cos\theta_3)\Big]$ $b_e(A) = \frac{m_1}{\pi}\Big[\frac{1}{2}(\cos 2\theta_3 - \cos 2\theta_2) + \frac{2}{A}(c\sin\theta_3 - b\sin\theta_2)\Big]$
L		K	$c=-a$ $b=a$	$g_e(A) = \frac{4B}{\pi A}\cos\theta_1$ $b_e(A) = -\frac{4B}{\pi A}\sin\theta_1$
M		H	$m_3=m_1$	$g_e(A) = \frac{1}{\pi}\Big[\pi m_1+(m_1-m_2)\Big\{(\theta_3-\theta_5) - \frac{1}{2}(\sin 2\theta_3 - \sin 2\theta_5) + \frac{2}{A}(c\cos\theta_3 - e\cos\theta_5)\Big\}\Big]$ $b_e(A) = \frac{(m_1-m_2)}{\pi}\Big[\frac{1}{2}(\cos 2\theta_3 - \cos 2\theta_5) + \frac{2}{A}(c\sin\theta_3 - e\sin\theta_5)\Big]$
N		M	$m_2=0$	$g_e(A) = \frac{m_1}{\pi}\Big[\pi+(\theta_3-\theta_5) - \frac{1}{2}(\sin 2\theta_3 - \sin 2\theta_5) + \frac{2}{A}(c\cos\theta_3 - e\cos\theta_5)\Big]$ $b_e(A) = \frac{m_1}{\pi}\Big[\frac{1}{2}(\cos 2\theta_3 - \cos 2\theta_5) + \frac{2}{A}(c\sin\theta_3 - e\sin\theta_5)\Big]$
O		N	$e=A$	$g_e(A) = \frac{2m_1}{\pi}\Big[\frac{\pi}{4} + \frac{\theta_3}{2} + \frac{\sin 2\theta_3}{4}\Big]$ $b_e(A) = -\frac{m_1}{\pi}\cos^2\theta_3$

TABLE II

NONLINEARITIES WITHOUT MEMORY

For all the nonlinearities without memory, $b_e(A)=0$

Alphabetical Classification	Characteristic	Modification of	Parameters Modified	Describing Function
P		A	$d=a$, $c=e=b$	$g_e(A)=\frac{1}{\pi}\Big[m_1\big(-2\theta_1+\sin 2\theta_1-4\frac{a}{A}\cos\theta_1\big)+(m_1-m_3)\big(2\theta_2-\sin 2\theta_2+4\frac{b}{A}\cos\theta_2\big)+\pi m_3\Big]$
Q		P	$m_3=0$	$g_e(A)=\frac{m_1}{\pi}\Big[2(\theta_2-\theta_1)+(\sin 2\theta_1-\sin 2\theta_2)+\frac{4}{A}(b\cos\theta_2-a\cos\theta_1)\Big]$
R		P	$b=a$	$g_e(A)=\frac{1}{\pi}\Big[m_3\{(\pi-2\theta_1)+\sin 2\theta_1\}-\frac{4}{A}(m_3a-B)\cos\theta_1\Big]$
S		R	$m_3=0$	$g_e(A)=\frac{4B}{\pi A}\cdot\cos\theta_1$
T		P	$a=0$	$g_e(A)=\frac{(m_1-m_3)}{\pi}\big(2\theta_2-\sin 2\theta_2+4\frac{b}{A}\cos\theta_2\big)+m_3$
U		T	$m_3=0$	$g_e(A)=\frac{m_1}{\pi}\Big[2\theta_2-\sin 2\theta_2+4\frac{b}{A}\cos\theta_2\Big]$
V		T	$m_1=0$	$g_e(A)=\frac{m_3}{\pi}\Big[\pi-\big(2\theta_2-\sin 2\theta_2+4\frac{b}{A}\cos\theta_2\big)\Big]$
W		R	$a=0$	$g_e(A)=m_3+\frac{4B}{\pi A}$
X		W	$m_3=0$	$g_e(A)=\frac{4B}{\pi A}$

Describing Function of the General Polynomial Type Nonlinearity

The input-output characteristic of a general polynomial type nonlinearity is described by the following equation. There is no loss of generality in assuming that n is odd in

$$y=f(x)=c_n\cdot x^n+c_{n-1}\cdot x^{n-2}\cdot|x|+c_{n-2}\cdot x^{n-2}+c_{n-3}\cdot x^{n-4}\cdot|x|+\cdots+c_2\cdot x\cdot|x|+c_1\cdot x. \quad (10)$$

The reason for making use of the absolute value signs in the right side of (10) is to make $f(x)$ an odd function with respect to x.

From the derivation in Appendix III, it is seen that the describing function of the polynomial type nonlinearity described by (10) is given by (11) and (12). Thus,

$$g_e(A)=\frac{2}{\sqrt{\pi}}\left[c_nA^{n-1}\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)}+c_{n-1}A^{n-2}\frac{\Gamma\left(\frac{n+1}{2}\right)}{\Gamma\left(\frac{n+2}{2}\right)}+c_{n-2}A^{n-3}\frac{\Gamma\left(\frac{n}{2}\right)}{\Gamma\left(\frac{n+1}{2}\right)}+\cdots+c_2A\frac{3\sqrt{\pi}}{4}+c_1\frac{\sqrt{\pi}}{2}\right] \quad (11)$$

$$b_e(A)=0. \quad (12)$$

Eq. (11) may be expressed in the following compact form

$$g_e(A)=\frac{2}{\sqrt{\pi}}\sum_{k=1}^{n}c_kA^{k-1}\frac{\Gamma\left(\frac{k+2}{2}\right)}{\Gamma\left(\frac{k+3}{2}\right)}. \quad (13)$$

In (11) and (13), Γ represents the gamma function.

A special case of the polynomial type of nonlinearity is the nonlinearity characterized by an integral power of the input. The input-output characteristic of this type of nonlinearity is defined by

$$y = f(x) = c \cdot x^n \quad \text{for } n \text{ odd} \tag{14}$$

$$y = f(x) = \begin{cases} c \cdot x^n & x \geq 0 \\ -c \cdot x^n & x < 0 \end{cases} \quad \text{for } n \text{ even.} \tag{15}$$

From the derivation in Appendix III, the describing function of either of the nonlinearities characterized by (14) and (15) is given by

$$g_e(A = \frac{2c}{\sqrt{\pi}} \cdot A^{n-1} \frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)} \tag{16}$$

$$b_e(A) = 0. \tag{17}$$

Special cases of (14) and (15) which occur frequently are the cubic nonlinearity defined by $y=x^3$ and the absquare nonlinearity defined by $y=x|x|$, respectively. Substituting $n=3$ in the right side of (16), it is seen that for the cubic nonlinearity, $g_e=\frac{3}{4} \cdot c \cdot A^2$. Similarly, substituting $n=2$ in the right side of (16), it is seen that for the absquare nonlinearity, $g_e(A)=$

$$\frac{8cA}{3\pi}.$$

Conclusions

The describing functions of two general types of nonlinearities have been derived. The two general types are called the piecewise linear type nonlinearity and the polynomial type nonlinearity. A large number of nonlinearities which occur in practice are shown to be modifications of these general types of nonlinearities. The piecewise linear type nonlinearities are divided into two groups, *viz.*, nonmemory type and memory type devices. The describing functions of all the piecewise linear type nonlinearities are tabulated.

The plots of the describing functions have not been included here since it is not possible to draw general plots for most of the cases considered. The describing function plots for some of the simpler types of nonlinearities are drawn in Goldfarb's paper.[1] In the more complicated types of nonlinearities, there are too many parameters associated with the nonlinearity so that the describing function plots will consist of families of curves. It is felt that it will be simpler to evaluate the describing function for any nonlinearity with a particular set of parameters by substituting the values of the parameters in the formulas derived rather than by drawing whole families of curves for each nonlinearity.

Appendix I

Some Relations Associated with the General Piecewise Linear Type Nonlinearity Shown in Fig. 1

The input-output relationship of the general piecewise linear type nonlinearity shown in Fig. 1 may be expressed as follows.

$$y = 0 \qquad -d < x < a; \quad \dot{x} > 0 \tag{18}$$

$$y = m_1(x-a) \qquad a \leq x < b; \quad \dot{x} > 0 \tag{19}$$

$$y = m_3\left[x - \left(b - \frac{B}{m_3}\right)\right] \qquad x \geq b \tag{20}$$

$$y = m_1(x-a) \qquad e \leq x < b; \quad \dot{x} < 0 \tag{21}$$

$$y = m_2\left[x - \left(\frac{Dc - Ee}{D - E}\right)\right] \qquad c \leq x < e; \quad \dot{x} < 0 \tag{22}$$

$$y = m_1(x-d) \qquad d \leq x < c; \quad \dot{x} < 0 \tag{23}$$

$$y = 0 \qquad -a \leq x < d \tag{24}$$

$$y = m_1(x+a) \qquad -b \leq x < -a; \ \dot{x} < 0 \tag{25}$$

$$y = m_3\left[x + \left(b - \frac{B}{m_3}\right)\right] \qquad x < -b \tag{26}$$

$$y = m_1(x+a) \qquad -b \leq x \leq -e; \ \dot{x} > 0 \tag{27}$$

$$y = m_2\left[x + \left(\frac{Dc - Ee}{D - E}\right)\right] \qquad -e \leq x < -c; \ \dot{x} > 0 \tag{28}$$

$$y = m_1(x+d) \qquad -c \leq x < -d; \ \dot{x} > 0. \tag{29}$$

Let the input to the nonlinear element be x where

$$x = A \sin \theta. \tag{30}$$

Here

$$\theta = \omega t \tag{31}$$

where ω is the frequency of the sine wave.

Referring to Fig. 1, it is seen that

$$\theta_1 = \arcsin \frac{a}{A} \tag{32}$$

$$\theta_2 = \arcsin \frac{b}{A} \tag{33}$$

$$\theta_3 = \arcsin \frac{c}{A} \tag{34}$$

$$\theta_4 = \arcsin \frac{d}{A} \tag{35}$$

$$\theta_5 = \arcsin \frac{e}{A}. \tag{36}$$

Here θ_1, θ_2, θ_3, θ_4, and θ_5 are associated with the corners of the input-output characteristic shown in Fig. 1.

Appendix II

Development of Describing Function Equations for the General Piecewise Linear Type Nonlinearity

In this section the right side of (4) will be evaluated. Substituting for $f(A \sin \theta)$ from (18) to (29) in (4) yields

$$J_{n1}(A) = \frac{j}{\pi A}\left[\int_{\theta_1}^{\theta_2} m_1(A \sin \theta - a) \cdot \exp(-j\theta) \cdot d\theta\right.$$

$$+ \int_{\theta_2}^{\pi-\theta_2} m_3(A \sin \theta - k_3) \cdot \exp(-j\theta) d\theta$$

$$+\int_{\pi-\theta_2}^{\pi-\theta_5} m_1(A\sin\theta - a)\cdot\exp(-j\theta)d\theta$$

$$+\int_{\pi-\theta_5}^{\pi-\theta_3} m_2(A\sin\theta - k_2)\cdot\exp(-j\theta)d\theta$$

$$+\int_{\pi-\theta_3}^{\pi-\theta_4} m_1(A\sin\theta - d)\cdot\exp(-j\theta)d\theta$$

$$+\int_{\pi+\theta_1}^{\pi+\theta_2} m_1(A\sin\theta + a)\cdot\exp(-j\theta)d\theta$$

$$+\int_{\pi+\theta_2}^{2\pi-\theta_2} m_3(A\sin\theta + k_3)\cdot\exp(-j\theta)d\theta$$

$$+\int_{2\pi-\theta_2}^{2\pi-\theta_5} m_1(A\sin\theta + a)\cdot\exp(-j\theta)d\theta$$

$$+\int_{2\pi-\theta_5}^{2\pi-\theta_3} m_2(A\sin\theta + k_2)\cdot\exp(-j\theta)d\theta$$

$$+\int_{2\pi-\theta_3}^{2\pi-\theta_4} m_1(A\sin\theta + d)\cdot\exp(-j\theta)\cdot d\theta\Big]. \quad (37)$$

In (37), k_2 and k_3 are defined as the following quantities:

$$k_2 = \frac{Dc - Ee}{D - E} \quad (38)$$

$$k_3 = b - \frac{B}{m_3}\cdot \quad (39)$$

Evaluating the first integral in the right side of (37),

$$\int_{\theta_1}^{\theta_2} m_1(A\sin\theta - a)\cdot\exp(-j\theta)d\theta$$

$$= m_1\int_{\theta_1}^{\theta_2}\left\{\frac{A}{2j}[1-\exp(-2j\theta)] - a\exp(-j\theta)\right\}d\theta$$

$$= m_1\left\{\frac{A}{2j}\left[\theta + \frac{1}{2j}\exp(-2j\theta)\right] + \frac{a}{j}\exp(-j\theta)\right\}\Bigg|_{\theta_1}^{\theta_2}$$

$$= \frac{m_1}{j}\left\{\frac{A}{2}\left[(\theta_2-\theta_1) + \frac{1}{2j}(\exp\overline{-2j\theta_2} - \exp\overline{-2j\theta_1})\right]\right.$$

$$\left. + a(\exp\overline{-j\theta_2} - \exp\overline{-j\theta_1})\right\}. \quad (40$$

Similarly the other integrals in the right side of (37) may be evaluated and (37) solved to evaluate $J_{n1}(A)$. Separating the real and imaginary parts of $J_{n1}(A)$ yields (5) and (6).

Appendix III

Development of the Describing Function Equations for the General Polynomial Type Nonlinearity

In deriving the describing function of the general polynomial type nonlinearity defined by (10), each term of the polynomial will be considered separately. Thus, for purposes of derivation two types of nonlinearities will be assumed. There is no loss of generality by this assumption since each term in the polynomial will belong to one of the two types defined by (14) and (15); these two types will be considered separately.

Case 1—n Odd

Substituting from (14) in (2) and (3) yields

$$g_e(A) = \frac{1}{\pi A}\int_0^{2\pi} c(A\sin\theta)^n\cdot\sin\theta\cdot d\theta$$

$$= \frac{c}{\pi A}\cdot A^n\int_0^{2\pi}\sin^{n+1}\theta\cdot d\theta$$

$$= \frac{4c}{\pi}\cdot A^{n-1}\int_0^{\pi/2}\sin^{n+1}\theta\cdot d\theta$$

$$= \frac{2c}{\sqrt{\pi}}\cdot A^{n-1}\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)}\cdot \quad (41)$$

Here, Γ denotes the gamma function

$$b_e(A) = \frac{1}{\pi A}\int_0^{2\pi} c(A\sin\theta)^n\cos\theta\cdot d\theta = 0. \quad (42)$$

Case 2—n Even

Substituting from (15) in (2) and (3) yields

$$g_e(A) = \frac{1}{\pi A}\left[\int_0^{\pi} c(A\sin\theta)^n\cdot\sin\theta\cdot d\theta\right.$$

$$\left. - \int_{\pi}^{2\pi} c(A\sin\theta)^n\cdot\sin\theta\cdot d\theta\right]$$

$$= A^n\frac{c}{\pi A}\left[\int_0^{\pi}\sin^{n+1}\theta\cdot d\theta - \int_{\pi}^{2\pi}\sin^{n+1}\theta\cdot d\theta\right]$$

$$= \frac{4c}{\pi}A^{n-1}\int_0^{\pi/2}\sin^{n+1}\theta\cdot d\theta$$

$$= \frac{2c}{\sqrt{\pi}}\cdot A^{n-1}\frac{\Gamma\left(\frac{n+2}{2}\right)}{\Gamma\left(\frac{n+3}{2}\right)}\cdot \quad (43)$$

$$g_e(A) = \frac{1}{\pi A}\left[\int_0^{\pi} c(A\sin\theta)^n\cdot\cos\theta\cdot d\theta\right.$$

$$\left. - \int_{\pi}^{2\pi} c(A\sin\theta)^n\cdot\cos\theta\cdot d\theta\right]$$

$$= 0. \quad (44)$$

Acknowledgment

The author is grateful to Dr. J. E. Gibson, professor of electrical engineering at Purdue University, for his extremely helpful criticism of this discussion.

Soviet Literature on Control Systems*

P. L. SIMMONS† AND H. A. PAPPO†

Summary—**Most of the better known articles from Soviet literature on the subject of control systems published between January, 1953 and March, 1959 are cited. Emphasis is on the theoretical aspects of control systems.**

The items are arranged alphabetically according to author, with no attempt at classification. The first section includes all topics and the second section lists only articles which have English translations. Annotations are given only for articles which the bibliographer has examined.

The introduction mentions some of the existing bibliographies in the field and discusses possible schemes for classifying bibliographic references to control systems.

JUST as a control device utilizes its sense apparatus to determine the nature of the control that it is to apply, so must control process analysts be aware of parallel investigations in the area if they are to maintain some semblance of control over the inquiry. To aid in the search for and to provide easy contact with material in the field, several special bibliographies have been prepared in recent years. So far, these efforts (the present one included) have attempted only limited surveys of the literature. At some time in the future it will obviously be in order to combine the various results in a single reference list. Even in the special lists, the source material is confined primarily to work done in this country and does not consider developments in the U.S.S.R. Since the products of Russian science are of increasing interest to us, an effort is made in this report to supplement the existing bibliographies with reference to that work.

In addition to the concentration on Russian literature, the present reference list differs in other respects from the existing bibliographies. No effort has been made to classify the items according to type of control system. Use of a simple alphabetical listing is justified by the fact that there are alternative classification schemes and that the selection of one scheme over another is a function of purposes which are too specialized for this bibliography. For instance, we might have classified the control systems according to the uses to which they are put. Clearly, though, the number of uses is extremely large. If we employed all of them in aggregating the reference items, the search time would be exorbitant, and we have no solid basis for choosing any subset of them. Again, a classification according to structure might have been used, but there too, there are many structural properties, and we have no sound reason to favor any one over another.

A theory of control systems which has been stated in rudimentary form in another paper[1] does in fact mitigate against a breakdown of control devices into permanent classes of these types. In that report an attempt was made to show that all control systems can be brought under a single principle. According to that theory each control device, from the very simplest feedback control unit through the varieties of extremal or optimal control systems or sampled-data systems to any one of the mnay kinds of adaptive control systems, can be seen as an element in a spectrum of control systems. If the theory is sound, it justifies neglecting as categories many of the terms which are found frequently in the literature and which are most often used in classification schemes. In fact, the continued use of these items for classification tends to gloss over strong similarities among control devices and creates pseudo or irrelevant differences.

Since an increasing number of articles are appearing in the field of control systems it becomes apparent that some mode of breaking down any list of references into manageable groups is imperative. This mode must not be such as to ignore important similarities or differences, however. Rather it must highlight the unifying principles where they occur, and it must put into sharp relief those features of control devices which are different enough to warrant the contrast. Any gains made in this direction will tend to accelerate the flow of information and to increase our knowledge of control processes.

In this country the area of adaptive control systems has been well covered by Aseltine, Mancini, and Sarture; they survey the field, give definitions, and provide a useful guide in a table of "Characteristics of Adaptive Systems."[2] Recently, Stromer has covered the literature in an annotated bibliography.[3] Another area which Stromer has documented is sampled-data systems; his selective bibliography[4] is a good reference for the period 1955–1959.

To our knowledge there does not exist a bibliography of Russian documentation in the field of control systems.

* Manuscript received by the PGAC, February 1, 1960. Published by System Development Corp., Santa Monica, Calif., SP-125; October 29, 1959.

† System Development Corp., Santa Monica, Calif.

[1] H. Pappo, "Theory of System Organization—1: Introduction to Control Systems," System Development Corp., Santa Monica, Calif., SP-120; October, 1959. The second document in this series is "Theory of System Organization—2: Preliminary Formulation," SP-121; October, 1959.

[2] J. A. Aseltine, A. R. Mancini, and C. W. Sarture. "A survey of adaptive control systems," IRE TRANS. ON AUTOMATIC CONTROL, no. PGAC-6, pp. 102–108; December, 1958. See p. 106.

[3] P. R. Stromer, "Adaptive or self-optimizing control systems—a bibliography," IRE TRANS. ON AUTOMATIC CONTROL, vol. AC-4, pp. 65–68; May, 1959.

[4] P. R. Stromer, "A selective bibliography on sampled-data systems," IRE TRANS. ON AUTOMATIC CONTROL, no. PGAC-6, pp. 112-114; December, 1958.

Before a comprehensive bibliography can be prepared, this gap must be filled.

If we can assume that Russian documentation in control systems keeps pace with her automation of production techniques, the number of documents should be large. Newton cites statistics to the effect that annually the Soviet Union invests 27 per cent of its gross national product in new plants compared with 19 per cent in the United States. In order to maintain its high rate of expansion, automation is keenly emphasized.[5] The fields in which we are interested—sampled-data systems, optimal or extremal systems (minimal or maximal), and adaptive systems—have been actively developed by the Academy of Science of the U.S.S.R. and the Institute of Automation and Telemechanics, the two principal Soviet organizations researching the area. Since 1953 the Institute has explored the area of sampled-data systems and optimal control systems with great vigor. The Institute's journal, *Automation and Remote Control* (*Automatika i telemekhanika*), enjoys high repute throughout the engineering world; proof of this lies in the fact that the National Science Foundation of the U.S.A. has made a grant to the Instrument Society of America for translating this publication into English. An outstanding individual in the institute is Prof. J. Tsypkin, who is mainly interested in sampled and quantitized data systems. Another prominent investigator is Pugachev, who has worked on the analysis of systems subjected to random disturbances.

In an effort to keep track of this literature, the following reference list has been compiled. The bibliography cites most of the better known articles for the period January, 1953 to March, 1959. Emphasis is on the theoretical aspects of control systems.

The first section of the bibliography, arranged by primary author, includes all topics. The second section lists only articles which have English translations. Annotations are given only for articles which the bibliographer has examined.

As a final comment, it should be pointed out that a fairly large class of articles has been excluded from this list. These articles, for the most part, deal with very special problems in the application of control systems, and emphasize the electronics of control systems rather than theory. For this reason it was felt that they did not properly belong in the list. Persons who are interested in these details may consult the Library of Congress *Current List of Russian Accessions*, issued monthly.

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Volchkov, K. S. "Some optimal relationships in ideal and controlled magnetic amplifiers," *Avtomatika i telemekhanika* (*Automation and Remote Control*), vol. 19, pp. 85–95; January, 1958.

Zalmanzon, L. A. "Some aspects of pneumatic controller design," *Avtomatika i telemekhanika* (*Automation and Remote Control*), vol. 18, pp. 87–91; January, 1957.

In English Translation

Andreev, N. I. "Definition of an optimal linear dynamic system from an extremum criterion applied to a particular form of functional," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 18, pp. 665–670; July, 1957.

A method is proposed for determining the dynamic system which corresponds to an extremum of a functional which can be represented as a differential function of several square functionals of the unit transient function of the systems. The necessity for the function to have an extremum is deduced in this paper.

Andreev, N. I. "A theory for determining optimum dynamic systems," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 1049–1062; December, 1958.

Batkov, A. M. and V. V. Solodovnikov. "Method for determining the optimum characteristics of a certain class of self-adaptive control system," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 18, pp. 411–428; May, 1957.

The problem of optimalizing a system, in the sense of reducing the sum of the squares of the dynamic and mean-square deviation to a minimum is considered for one class of linear systems with variable parameters when a stationary level of random noise is present. This method gives systems which are optimal in that they are self-adaptive to the nature of the signal input.

Chugin, IU. I. "Optimal frequency deviation in one channel telemetering system," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 339–348; April, 1958.

A method is given of calculating the optimum frequency deviation in a single-channel telemetering system with fluctuating noise. The method is an analysis of the energy spectrum of the noise.

Da-Chuan, S. "On the possibiltiy of a certain type of oscillation in sampled-data control systems," *Automation Express* (*Avtomatika i telemekhanika*), vol. 1, pp. 11–14; May, 1959.

This is an abstract only, not a complete translation.

Druzhinin, V. G. "On the number of reserve sections," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 1035–1038; November, 1958.

Selection of the optimum number of reserve sections in any given automatic control system is examined.

Ermakov, S. S. and E. M. Esipovich. "A way of forming transfer functions of sampled-data control systems with extrapolating devices," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 395–401; May, 1958.

This paper deals with a way of forming conditional transfer functions of extrapolating devices for analog-to-digital conversion. The transfer function expression depends on the shape of the input pulses.

Fan, C. W. "Concerning a method for analyzing sampled-data systems," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 288–299; April, 1958.

Strelkov's method, well known in Russia, for analyzing continuous linear systems, is generalized on the basis of a discrete Laplace transform and applied to the qualitative evaluation of the ripple in sampled-data systems.

Fel'baum, A. A. "Automatic optimalizer," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 718–728; August, 1958.

Deals with the problem of constructing a machine which minimizes a function of several variables in the presence of additional restrictions. The circuits of different models are represented.

Fitsner, L. N. "Choice of a power unit for an optimum automatic control system," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 1077–1086; December, 1958.

Fitsner, L. N. "On selecting the power section for optimal automatic control systems," *Automation Express* (*Avtomatika i telemekhanika*), vol. 1, p. 8; April, 1959.

This is an abstract only.

Gamkrelidze, R. V. "On the general theory of optimal processes," *Automation Express* (*Akademiia nauk SSSR Doklady*), vol. 1, pp. 37–39; April, 1959.

This is an abstract only.

Gamkrelidze, R. V. "The theory of optimum speed of responses in linear systems," *Automation Express* (*Akademiia nauk SSSR Izvestia. Seria matem.*), vol. 1, pp. 38–40; January, 1959.

This is an abstract only.

Gorskii, V. V. "A relay automatic position control system using a compound-wound motor," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 18, pp. 832–842; September, 1957.

A relay automatic position control system using a compound-wound system using an actuator is considered. The formulations for determining the output shaft speed and rotation angle during follow-up are deduced. The resulting relay system is considered to be close to optimal.

Kilinski, A. "Measure of the reliability of electronic equipment," MDF. K-185, Washington, D. C. (Translation of *Przeglad telekamumikacyjny*, vol. 31, no. 25, pp. 200–207; 1958). 21 pp. 1959.

The most common definition of the reliability of "best" result is discussed in terms of probability, failure, time between failure, etc. Based for the most part on American periodical sources. (This translation can be ordered from the translator: Morris D. Friedman, Inc., P. O. Box 35, Newton 65, Mass.).

Krasovskii, N. "On the theory of optimal regulation," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 18, pp. 1005–1016; November, 1957.

Effective methods developed in an earlier paper are generalized for those cases. Formulation described above takes a different approach to this problem. Gives results of works by Fel'baum, Pontriagin, and Boltianski. Presents conditions for existence of optimum trajectories.

Krug, E. K. and O. M. Minina. "Optimal transients in an automatic control systems having a bounded regulator unit," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 8–21; January, 1958.

The forms of the optimum transients in automatic control systems are determined for those with objects of various response characteristics, assuming the control valve to be position bounded. It is shown that there are difficulties in using continuous-action regulators to produce optimal transients, because of nonlinear converters.

Kurakin, K. I. "Analytical method of synthesis of linear control systems in the presence of interference and with a specified dynamic accuracy," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 402–412; May, 1958.

An analytical method of approximation of transcendental transfer functions of automatic control systems is proposed. The functions were obtained with the help of fraction-rational functions to be determined by optimum control characteristics. The practical applications are outlined by example.

Kurakin, K. I. "The design of linear follower systems from the criterion of minimum practical limiting reproduction error," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 18, pp. 447–465; May, 1957.

The optimum transfer function of a servo system is determined for the case where the input to be reproduced is a known slowly-varying function of time and the noise spectrum is uniform over the working frequency range. The following design is based on the criterion of minimum practical follow-up error given previously by this author. Methods of producing an optimum servo-transfer are described which employ a corrector unit operating on dc.

Letov, A. M. "Conditionally stable controlled systems concerning a certain type of optimum controlled system," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 18, pp. 649–664; July, 1957.

Attempts to show mathematically that a strictly linear system designed to accord with any preconceived criterion of optimum quality can be altered to provide damping within the first swing by converting this system to one which is conditionally stable.

Morosanov, I. S. "Conference on the theory and application of discrete automatic control systems," *Automation Express* (*Akademiia nauk SSSR. Izvestia. Otdelinia tehn. Energetika i avtomatika*), vol. 1, pp. 10–11; May, 1959.

This is an abstract only. The conference covered sampled-data systems and relay systems; there were also 13 papers devoted to the theory of self-adaptive systems, the principles of extremal systems, and the description of equipment which has been tested.

Morosanov, I. S. "Methods of extremum control," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 18, pp. 1077–1092; November, 1957.

The main methods of extremum control are considered. Systems are classified according to the method of seeking the optimum. Calculation of self-oscillation modes is demonstrated using a relay system as an example. Suggestions as to the practical applications are given.

Morosanov, I. S. and P. I. Chinayev. "Conference on the Theory and Application of Discrete Automatic Systems," Joint Publications Research Service, Washington 25, D. C., Series L-731-37 (Translation of *Avtomatika i telemekhanika*, vol. 20, pp. 100–106; 1959); April 14, 1959.

Oldenburger, R. "Stabilizing control systems with a special signal," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 18, pp. 429–433; May, 1957.

Linear automatic systems with two or more main delay links may be stabilized by injecting noise or a signal of sufficiently high frequency subject to the condition that system hunt is not too severe. The signal amplitude must be sufficient to drive a bounded element to its limits, *e.g.*, must correspond to the control range of an auxiliary control valve in the regulator. Optimal nonlinear control systems with one or more delay links can also be stabilized by injecting a suitable signal.

Ostrovskii, IU. I. "Extremum regulation," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 18, pp. 900–907; September, 1957.

The extremum controller is described in the Soviet and foreign (English) literature and the main designs are presented. A classification by type is given.

Ostrovskii, IU. I. "Pneumatic extremum regulator," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 18, pp. 1093–1098; November, 1957.

A pneumatic extremum regulator developed at the Institute of Automation and Remote Control, Academy of Sciences of the USSR, is described. Laboratory results are given also.

Perov, V. P. "Synthesis of pulse networks and systems with pulse feedback," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 18, pp. 1129–1145; December, 1957.

The optimum characteristics of pulse systems and their realizations are determined. The optimum criterion adopted is the condition that the error have a minimum dispersion for a specified dynamic accuracy and for a specified system transient time. It is assumed that the driving function consists of noise and a signal and that the noise is a stationary random function, while the signal is the sum of a random component.

Pugachev, V. S. "The determination of an optimal system by some arbitrary criterion," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 513–532; June, 1958.

The method is expounded for the determination of an optimal system by some arbitrary criterion of the Bayes type, out of the class of all functions to which this can be meaningfully applied. The solution of the principle of determining the operator of the optimal system is then reduced to the finding of certain linear operators of the minimum of some function or functionals.

Pugachev, V. S. "Integral canonical representations of random functions and their application in deriving optimal linear systems," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 18, pp. 1017–1031; November, 1957.

Theory of integral canonical representations of the random functions and their use in determining the optimal linear operator is given. Use of this method leads to a formula for the weighting function of an optimal one-dimensional linear system for the case where the observation interval is unlimited when the observed random function is a result of passing white noise through some system.

Rozeman, E. A. "On the limiting speed of action of servo-system with power, moment and rate limitations on the executive elements," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 622–644; July, 1958.

The problem of the shortest transient response is considered for servo-systems with power, moment and rate limitations on the executive elements. The phase plane of the system states is subdivided with respect to the character of the optimal transient response for the stated limitations. From the solution of the variation problem, the form of the shortest response is then determined.

Rozeman, E. A. "Optimum transients in saturating systems," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 18, pp. 539–555; June, 1957.

Solves the problem of determining the nature of the shortest transient response in power-saturating systems. It is demonstrated that the optimum low for the variation of the actuating-motor current is almost linear for large heating time constants. A family of isochrone regions is plotted, and a comparison is made between optimum and current saturation.

Shigin, E. K. "Synthesis of automatic control systems with a sign-changing input for the integrating corrective element," *Automation Express* (*Nauchnye Doklady vysshei shkoly, Mashinostroenie i priborostroenie*), vol. 1, pp. 3–6; April, 1959.

Complete English translation. Discusses the last decade of work on automatic control systems theory which has covered optimal transients responses and self-adaptive systems.

Smyrova, N. A. "Additions to the table of optimal characteristics given by Solodovnikov and Matveev," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 370–372; April, 1958.

An extension of tables of optimum characteristics is given here for corrective servo systems.

Stakhovskii, R. I. "Twin-channel automatic optimalizer," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 729–740; August, 1958.

The circuit and some units of a twin-channel electronic automatic optimalizer for the location of minima is described in detail. Experimental results are correspondingly given for its modes.

Tsypkin, IU. Z. "Sampled-data systems with extrapolating devices," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 383–394; May, 1958.

Automatic sampled-data systems containing extrapolating devices are investigated. Equations for such systems, describing the process at any given moment of time, are given, and the method of analyzing such systems is illustrated by examples.

Velershtein, R. A. and A. A. Fel'baum. "Development of an almost optimal system by means of an electronic analog," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 808–818; September, 1958.

Velershtein, R. A. and A. A. Fel'baum. "Using an electronic simulator to formulate the circuit of an almost optimum system," *Automation Express* (*Avtomatika i telemekhanika*), vol. 1, pp. 40–41; January, 1959.

Volchkov, K. S. "Some optimal relationships in an ideal ac controlled magnetic amplifier," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 19, pp. 76–85; January, 1958.

The functioning of an ideal ac controlled, active load, choked magnetic amplifier is investigated. Optimal relationships are deduced for amplifiers designed either for amplifying signals of one fixed frequency or for comparative purposes.

Zalmanzon, L. A. "Some aspects of pneumatic controller design," *Automation and Remote Control* (*Avtomatika i telemekhanika*), vol. 18, pp. 93–98; January, 1957.

Conditions are indicated under which it is desirable to supplement controllers of the normal type of automatic adjustment to the maximum or minimum of some quantity dependent on the controlled parameter. One possible way of doing this, by means of pneumatic devices, is considered; automatic adjustment is performed by a device which consists of a divider coupled to a system of parallels and intersecting pipes, together with certain other units.

Correspondence

Adaptive and Optimalizing Control Systems*

In recent literature new control concepts have been introduced under many different names: adaptive, optimalizing, identifying, self-optimizing, etc. Scanning these papers[1] reveals that there is much confusion with respect to definitions and ideas, while a basic philosophy on these concepts is still almost completely lacking.

The confusion may be partly due to the fact that the term "adaptive" has been taken from biology[2] where it already has a double meaning:[3] race—maintaining behavior by natural selection; individual organism—maintaining behavior by learned reactions.

Quoting some attempts to give definitions may be illustrative at this point. A general definition is:[4]

Adaptation

A process of fitting, or modifying a thing to other uses and so altering its form or original purpose. . . . In biology, adaptation plays a prominent part as the process by which an organism or species becomes modified to suit the conditions of life. Every change in a living organism involves adaptation; for in all cases life consists in a continuous adjustment of internal to external relations. Some adaptations are produced afresh in each generation, others are transmitted by heredity, having been probably fixed by natural selection.

Four engineering-system definitions are:[5–8]

—Adaptive control is a method of control aimed at obtaining optimum system performance even when there exists incomplete or inexact analytical or analog model of the process that is being controlled.
—The term "adaptive" will be applied to any control system which measures, continuously or intermittently, the impulse response or some other function which characterizes the system and which makes use of this system-characteristic function to determine and to generate the necessary forcing function to cause the system to behave in a desired manner.
—A feedback control system is adaptive if the sensitivity with respect to a variable x is zero over an interval in x of nonzero magnitude.
—An adaptive system is any physical system which has been designed with an adaptive viewpoint.

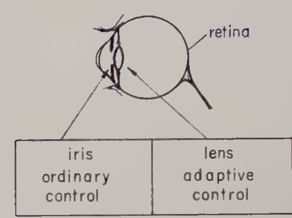
Due to the many different schemes that have been proposed already under names such as adaptive, optimalizing, etc., it will be difficult to reach agreement with respect to a definition on a topological basis. This, however, is not too important at present. The thing that is needed mostly today is the recognition of the basic differences between the "ordinary-control" and these newer control concepts. On this, perhaps, a general theory can be based.

"Odd-Function" and "Even-Function" Error

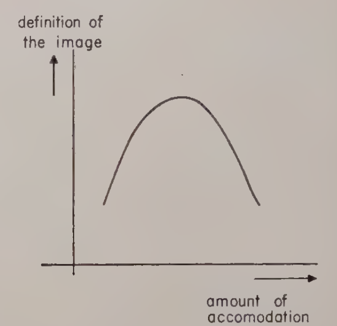
Let us have a look in the biological field, from which the term "adaptive" has been borrowed. The human eye is a very fine example because it shows both ordinary and adaptive types of control [Fig. 1(a)].

The *iris* controls the illumination level of the retina in an ordinary control system way; if the level is too high, then the error signal causes the iris to contract and vice versa. The sign of the error signal is a direct indication of the direction of iris change. The error signal goes to zero.

The *lens* controls the definition of the image on the retina. Now there is no error signal that changes sign when the lens accommodates from too far a distance to too close or vice versa. The function of definition vs amount of accommodation, and so the error signal, has an extremum [Fig. 1(b)].



(a)



(b)

Fig. 1—(a) The human eye; two types of control. (b) Lens adaptation; "even-function" error.

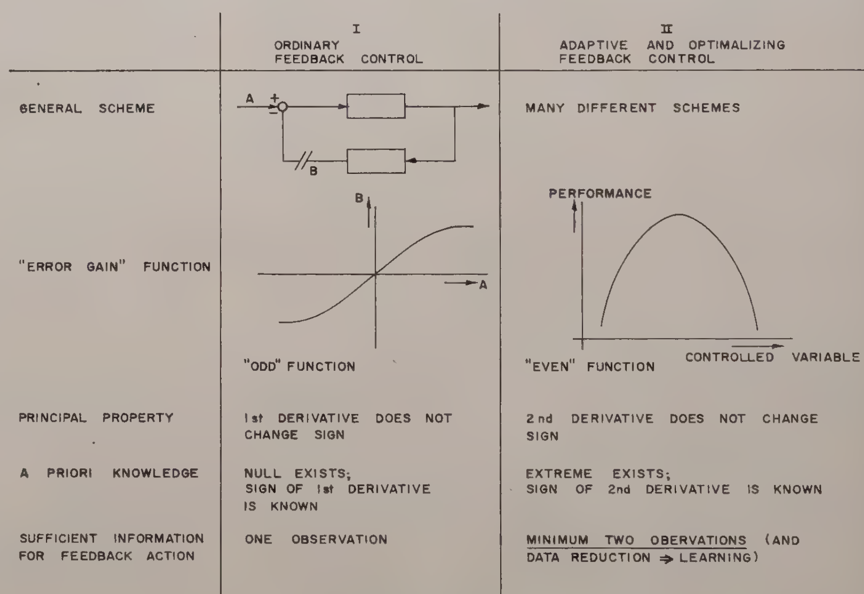


Fig. 2—Comparison of "odd-function" and "even-function" error control.

Now the proposition can be stated that all adaptive and optimalizing feedback control systems have essentially the characteristics of this eye-lens control. Fig. 2 gives a comparison with ordinary feedback control.

For the ordinary feedback the *loop gain* for the error signal is considered, in view of nonlinear computer-servo applications. Until a better word is found, the two types of feedback systems will be called, with respect to the fundamental difference, "*odd-function*" (error) systems, and "*even-function*" (error) systems.

From Fig. 3(b), it is self-evident that in the case of adaptive and optimalizing feedback control, when the sign of the second derivative is known (maximum or minimum of the function), at least two observations are required to locate the direction of the extremum with respect to these observations.

In order to perform control action we have to reduce the information of the two or more observations to one quantity. This process of data reduction could be called a "*learning*" process. *It changes the "even-func-*

* Received by the PGAC, February 16, 1960.

[1] Surveys of papers on these control concepts:
a) J. A. Aseltine, A. R. Mancini, and C. W. Sarture," A survey of adaptive control systems," IRE Trans. on Automatic Control, no. PGAC-6, pp. 102–108; December, 1958.
b) M. J. Levin, "Methods for the Realization of Self-Optimizing Systems," ISA, Paper No. FCS 2–58; April, 1958.
c) P. R. Stromer, "Adaptive or self-optimizing control systems; a bibliography," IRE Trans. on Automatic Control, vol. AC-4, pp. 65–68; May, 1959.

[2] R. F. Drenick and R. A. Shahbender, "Adaptive servomechanisms," *Trans. AIEE*, vol. 76, pt. II, pp. 286–292; November, 1957.

[3] W. R. Ashby, "Design for a Brain," John Wiley & Sons, Inc., New York, N. Y.; 1952.

[4] *Encyclopaedia Britannica*; 1957.

[5] C. F. Taylor, "Problems of nonlinearity in adaptive or self-optimizing systems," IRE Trans. on Automatic Control, no. PGAC-5, p. 66; July, 1958.

[6] L. Braun, "On adaptive control systems," 1959 IRE National Convention Record, pt. 4, pp. 32–44.

[7] E. Mishkin and R. A. Haddad, "Identification and command problems in adaptive systems," 1959 WESCON Convention Record, pt. 4, pp. 125–135. (Definition due to Truxal.)

[8] J. G. Truxal, "Trends in adaptive control," *Proc. Natl. Electronics Conf.*, vol. 15, pp. 1–16; October, 1959.

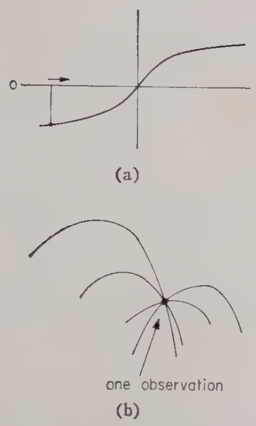


Fig. 3—(a) "Odd-function" error curve; one observation is sufficient to know the direction of the desired control action. (b) "Even-function" error curves; one observation is insufficient to give the direction of the desired control action.

tion" error into an "odd-function" error. After this process is applied, ordinary control principles can be used.

Definition

Based on these considerations we can propose another attempt at a definition:

> An adaptive or optimalizing system is a system which mathematically has to be described by differential equations with time-variable coefficients (vari-linear or vari-nonlinear equations). The time functions of the coefficients which we have under control are determined via some "process of learning" from the environment and/or the system characteristics in order to obtain an optimum behavior of the combination system+environment.

Part of the burden has been shifted to the "process of learning," so this has to be given some thought.

The Process of "Learning"

Two or more observations on the same variable can be made: at the same time in different systems; at different times in the same system. In general, the first possibility is less attractive, for it requires two or more identical processes. For guaranteeing the identity of several processes we need a knowledge of, *e.g.*, the physical/chemical theory and data which will surpass the available knowledge and the economically feasible instrumentation. In cases where an approximate model may be used for comparison this method may be quite feasible. In many cases, we have to resort to the second method to derive knowledge from the system.

The general scheme of "even-function" (error) control is given by:

measurement of instantaneous values
↓
"learning" (data reduction) ——— "even-function" error
↓
(decision)
↓
adjustment of system ——— "odd-function" error

The word "learning" is used in a general meaning of deriving one form of information from another by a given set of rules. The sense in which it will be used corresponds with the intuitive notions: that more than one observation is necessary; that observations at different time instants have to be related or averaged (not an instantaneous process). As the simplest engineering examples the following can be considered: squaring and averaging over time interval for rms error determination; multiplication of two signals and averaging over time interval for cross-correlation application.

Because we must work with signal characteristics instead of instantaneous values and we must combine observations made at different times, it is clear 1) that the process of learning requires time; 2) that the frequency spectrum of the knowledge resulting from the learning is narrower than that of the original signal(s). Therefore it is postulated that the time function of the differential equation coefficients in an engineering sense can be treated as if there were no connection of this time function with each of the instantaneous signal values. By this assumption the strictly-mathematically-speaking nonlinear system can be treated as a more tractable vari-linear system. In all well-known nonlinear systems the actual instantaneous values are used for the control action, so a clear distinction can be made between the intentionally-nonlinear and "even-function" (error) systems.

"Adaptive" and "Optimalizing"

A distinction between these types of systems can be found by regarding the input of external information to a system:

System	Input
1) Regulator	Constant value
2) Program-controlled	Predetermined variable values
3) Servo	Non-predetermined variable values
4) Implicit computer	
5) "Optimalizing plants"	No external information

In cases 1–4 we shall use the expression "adaptive" systems, if they satisfy the remarks with respect to an "even-function" error. In many cases there exists already a primary feedback loop; the adaptation of the system is done by a secondary feedback loop (Fig. 4). In case 5 we shall use "optimalizing" if the system satisfies the definition given above (Fig. 5).

In cases 1–4, the "conservation" of the amount of information while processing it to a higher power level or by the nonlinearity is the primary goal; *the adaptation tries to make the best of this information processing* by changes in the primary, already existing, feedback loop. In 5) there is no primary (over-all) feedback loop; optimalization introduces such a loop. The only information input was during the development of the system, *i.e.*, the knowledge that there exists an extremum in the process operation and in

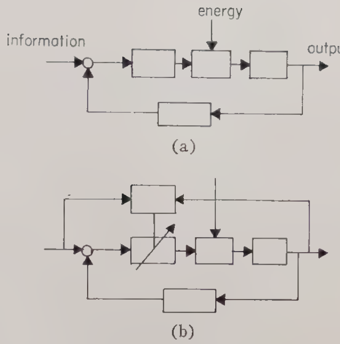


Fig. 4—Application of adaptive control. (a) Primary feedback loop. (b) Primary and secondary feedback loop: adaptive system.

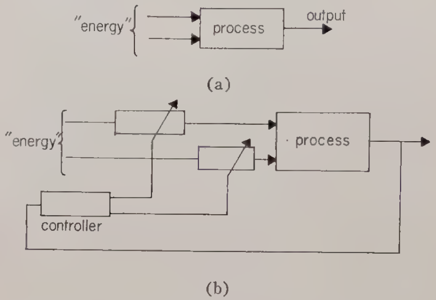


Fig. 5—Application of optimalizing control. (a) No primary feedback loop. (b) Optimalizing feedback loop.

what operating region. *The optimalization tries to make the best of the profit, energy, quality, or other commodity or economic value.*

Of course, the way of separation of system and environment is important in this respect. There may be primary (information carrying) feedback loops inside the plant, while the complete system may be an optimalizing one.

In recapitulation, the central ideas are: the process of learning; the optimum behavior with respect to: 1) information processing in adaptive systems, and 2) energy, profit or other economic value or commodity in optimalizing systems.

Purposes

The purposes of the new types of control are to give improvements, compared to ordinary feedback control or systems without feedback, in the sense of 1) counteracting the unpredictable changes in the system; 2) changing the system with regard to environmental conditions and with regard to variation of the class of input function, etc.; 3) compensating partly for component failure.

According to what quantity of the combination system+environment is used for the "learning process" we distinguish:

Adaptive (Information Processing Systems)

a) Input-Character (Change) Sensing
e.g., signal power to noise power level
signal spectrum to noise spectrum character
steady state or transient condition
probability of amplitude levels.

b) Load Disturbance-Character (Change) Sensing
   e.g., signal power to noise power level
   signal spectrum to noise spectrum character
   steady state or transient condition
   probability of amplitude levels.
c) Plant Parameter (Change) Sensing
   e.g., randomly varying parameters
   degree of stability.
d) Plant State (Change) Sensing
   e.g., energy storage, by which the dynamic behavior of the system is tied to history.
e) Performance Criterion Sensing (Information Processing of the Primary Feedback Loop)
   e.g., different error functions
   S/N level input to S/N level output.

The input-character and the load disturbance-character sensing types are feed forward systems; as soon as we monitor the output to study the effect of the adaptive changes we are using a performance criterion. As it is the case in all feed forward schemes, a thorough *a priori* knowledge of the system behavior is required. c), d), and e) are feedback systems.

Optimalizing (*Economic Values, Commodity Processing Systems*)

a) Environment Sensing
b) Process Extremum Sensing
   e.g., production rate
   purity of chemicals
   calculated profit
   actiradius (in transportation).

Here the environment sensing type is a feed forward system, the other a feedback arrangement.

Schematic Survey of Control Systems

The ideas developed in the foregoing give rise to the survey of Figs. 6 and 7. Adaptive systems are twice mentioned in these schemes due to the two feedback loops involved in these systems.

Concluding Remarks

The foregoing is a first attempt along a different line to get a fundamental idea about adaptive and optimalizing control systems. Of course it is neither complete nor free from defects.

The central points, however, seem to form a basis for discussion:

The need for some process of *"learning"* due to the *"even-function"* error or the dealing with signal characteristics instead of instantaneous values;

The distinction in adaptive and optimalizing systems according to the optimum sought for is in information processing or economic-commodity processing.

The definition proposed excludes the different types of pure nonlinear systems in which only instantaneous information without "learning" is used. Some of these systems are referred to as "passive adaptive."[1a]

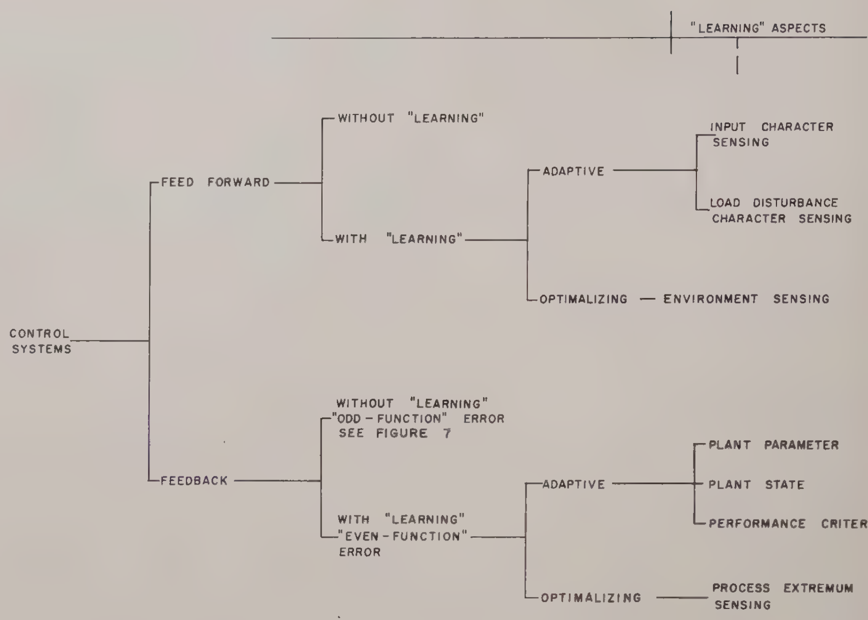


Fig. 6—Survey of control systems.

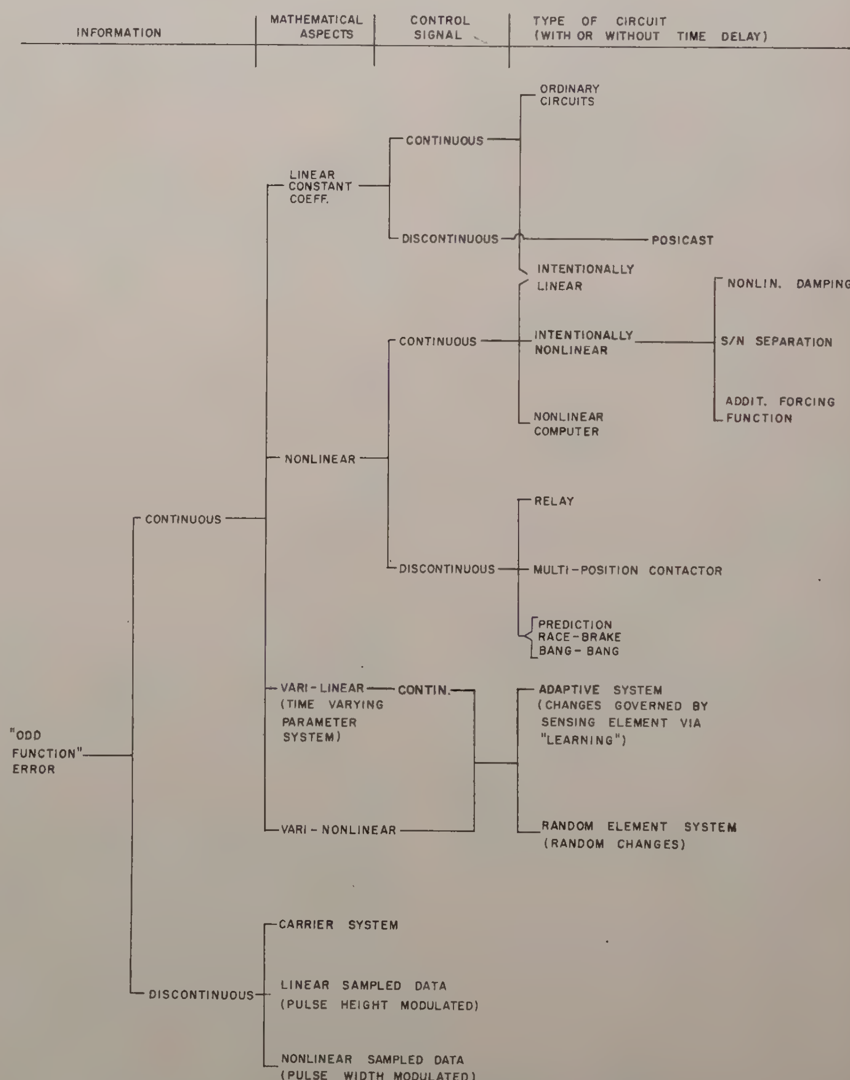


Fig. 7—Survey of control systems with "odd-function" erro

Should these be included then there is no reason for not including all nonlinear systems which give better response than linear systems.

The definition excludes also the interesting approach of Lang and Ham called "conditional feedback."[9] There feedback acts solely to reduce the influence of disturbances and thus determines the response of the system to external loads and internal parameter variations. So it has no more adaptive features than an ordinary feedback system; only there is more freedom in the construction, because the input/output requirements are taken care of by a model with the desired dynamic response. Of course this setup will be adaptive as soon as a parameter in the model or in the other part of the system is changed according to the statement given above.

It would be possible to define "levels of adaptation." An example:

Level	
0	No feedback
1	Ordinary feedback
2	Conditional feedback
3	Active adaptation by parameter changes
4	Active adaptation by parameter changes, with prediction of parameter changes from observations

The definition of active systems we should like to restrict to those having the levels 3, 4, $\cdots$.

The process of "learning" has to be studied in detail. The results of this "learning" will be values for the characteristics to be determined. These have to be expressed in terms of probability and accuracy (contaminating noise) and time. In general it might be possible to get as much information as is desired. However, this requires an excessive amount of time.

This leads to some fundamental questions:

What is the best way to derive information from the system with respect to learning and with respect to instrumentation (*e.g.*, data storage)?

What are the limitations to the speed and accuracy of learning with respect to the noise?

What is the relation of the learning process to the *a priori* knowledge about the system?

What are the basic limitations of adaptive and optimalizing control action?

What are the special requirements for a system to be suited to adaptation or optimalizing?

The development of these fascinating control concepts will depend on the answers to these types of questions.

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[9] G. Lang and J. M. Ham, "Conditional feedback systems—a new approach to feedback control," *Trans. AIEE*, vol. 74, pt. II, pp. 152–161; July, 1955.

[10] On leave from Technological University, Delft, The Netherlands, by an appointment supported by the International Cooperation Administration under the Visiting Research Scientists Program administered by the U. S. National Academy of Sciences.

A Note on Third-Order Linear Systems*

In feedback control system design, it is often useful to consider the correlation of the response to a step input with the closed-loop frequency response or zero-pole configuration. The usual situation is that certain pertinent time characteristics, such as rise time, per cent overshoot, and settling time, are predicted from certain characteristics of the frequency spectrum or the zero-pole locations. The correlations have been established exactly for a second-order system,[1] and are given approximately for high-order systems in certain cases.[2] The approximations generally depend upon the system being essentially described by a so-called "principal pair" of complex poles. If the remaining zeros and poles are "sufficiently far" from the principal pair, then the correlations based upon the second-order system hold with reasonable accuracy. Designers differ on what "sufficiently far" means, but generally if the distance from the origin to the principal poles is no greater than $\frac{1}{3}$ to $\frac{1}{5}$ the distance to the nearest of the other poles or zeros, the approximations are considered valid for design purposes, the aim here is to present the result of an analog computer study showing the effect on the rise time, per cent overshoot, and settling time to a step input as a pole on the negative real axis approaches the region of the complex plane which the principal pair of poles occupy.

The transfer function analyzed has the form $1/(1+2\zeta s+s^2)(1+\delta s)$, in which the time scale has been normalized, correspond- to an undamped natural radian frequency ω_0 equal to 1.0 rad/second. Depending upon the value of the damping constant ζ, the principal pair of poles lie somewhere on the unit circuit for $\zeta<1$. The parameter δ indicates the ratio of the radial distance of one of the complex poles (or the geometric mean of the principal poles if $\zeta>1$) to the radial distance of the real pole (see Fig. 1).

One of the design criteria is that the product of the rise time T_R (10 per cent to 90 per cent definition) and the upper half-power frequency f_2 (expressed in cps) is approximately equal to 0.35. Fig. 2 shows the variation of this product with increasing ζ for several different values of δ. The $\delta=0$ curve is thus for a second-order system. Note that as δ increases, the product T_Rf_2 generally becomes somewhat smaller for reasonably high values of ζ. For very low ζ, the product is very sensitive to ζ, particularly for large values of δ.

Fig. 3 shows the variation of per unit overshoot with M_p (the peak value of the frequency response) for various values of δ. Note that in the range of M_p from about 1.1 to 1.5, the per unit overshoot agrees approximately with the second-order system if δ does not exceed 0.3. For δ as large as 0.6, the variation is quite large, regardless of the value of M_p.

Fig. 4 shows the variation of settling time (in this case, time to settle to within 5 per cent of the final value) with ζ for various values of δ. In the intermediate ranges of ζ, the settling time increases by a factor ranging from 2 to 3 as the system changes from a second-order system where $\delta=0$ to a third-order system with $\delta=1.0$.

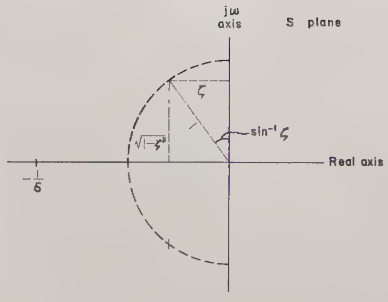

Fig. 1.

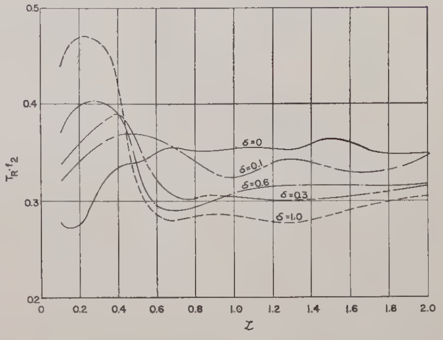

Fig. 2.

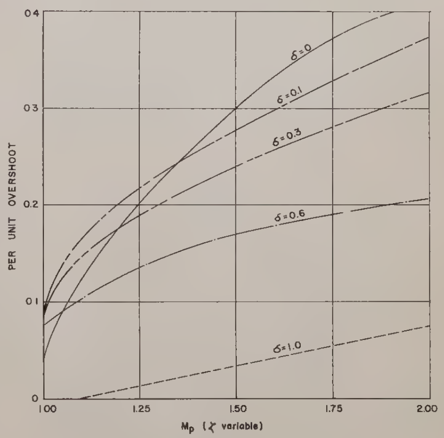

Fig. 3.

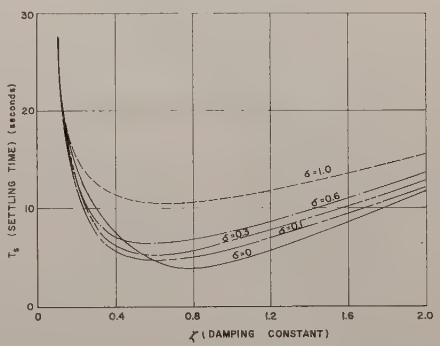

Fig. 4.

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* Received by the PGAC, December 18, 1959.

[1] For example, F. E. Nixon, "Principles of Automatic Controls," Prentice-Hall, Inc., Englewood Cliffs, N. J.; 1953.

[2] For example, M. E. Clynes, "Simple analytic method for linear feedback system dynamics," *Trans. AIEE* (*Applications and Industry*), vol. 75, pp. 377–383; January, 1956.

Improved Transient Response in Servo Systems with Input Modifications*

An ideal servo system should respond instantly with no overshoot. But due to its finite inertia, any system will require a finite time to reach the steady-state value and may overshoot the same if the effective damping is low (below critical damping). Introduction of large damping will eliminate the overshoot but the delay in response will increase to a large value. Quickest response is obtained if the damping coefficient is very small. However, a low damping causes large overshoots and damped oscillations occur for a number of cycles. It was, however, shown by Smith,[1] and also by Tallman and Smith,[2] that if we can modify the input signal appropriately it is possible to get a quick response with no overshoot. The input was split up into several fragments and was applied at such time intervals that the sum of all transients cancels out after the last excitation is applied. The same technique of input modification as that of Smith has been developed in this note with a different approach which is believed to be a more rigorous derivation of this method of control.

Let us consider a second-order servo system with effective mass m, damping coefficient R, and compliance D, to which a step input of amplitude a has been applied. The differential equation of such a system is given by

$$m\frac{d^2x(t)}{dt^2}+R\frac{dx(t)}{dt}+Dx(t)=a \qquad (1)$$

where $x(t)$ is the output at time t.

Taking Laplace transform and rearranging, we get

$$X(s)=\frac{s^2+a_1s+a_0}{s[(s+\alpha)^2+\beta^2]} \qquad (2)$$

where

$$a_1=\frac{R}{m}+\frac{x'(0)}{x(0)}$$

$$a_0=\frac{a}{mx(0)}$$

$x'(0)$ and $x(0)$ being initial velocity and displacement, respectively.

$$\alpha=\frac{R}{2m}=\text{the damping ratio}$$

$$\alpha^2+\beta^2=\frac{D}{m}=\omega_0^2 \text{ (say)}$$

(where $f_0=\omega_0/2\pi$ is the natural resonant frequency of the system);

$$\therefore\ \beta=\sqrt{\frac{D}{m}-\frac{R^2}{4m^2}}=\sqrt{\omega_0^2-\alpha^2}.$$

Taking inverse transform we get

$$x(t)=\frac{a_0}{\omega_0^2}x(0)+\frac{1}{\beta\omega_0}[(\alpha^2-\beta^2-a_1\alpha+a_0)^2+\beta^2(a_1-2\alpha)^2]^{1/2}e^{-\alpha t}\sin(\beta t+\psi) \qquad (3)$$

where

$$\psi=\tan^{-1}\frac{\beta(a_1-2\alpha)}{\alpha^2-\beta^2-a_1\alpha+a_0}+\tan^{-1}\frac{\beta}{-\alpha}.$$

The above equation shows that the output will attain its steady-state value $(a_0/\omega_0^2)x(0)$ for large values of t, when the second term in (3) is negligible. For low values of a, it will reach the steady-state value quickly but will overshoot the same and a number of damped oscillations will occur before the output settles to its final value $(a_0/\omega_0^2)x(0)$. However, for large values of α, β becomes imaginary and no oscillation (or overshoot) occurs. The output attains its steady-state value exponentially at a slow rate. However, if we can make the coefficient of the second term in (3) equal to zero, the output becomes independent of time and attains the steady-state value instantaneously. In order to do that, we must have

$$a_1-2\alpha=0 \text{ and } \alpha^2-\beta^2-a_1\alpha+a_0=0.$$

Now

$$a_1-2\alpha=\frac{R}{m}+\frac{x'(0)}{x(0)}-\frac{R}{m}=\frac{x'(0)}{x(0)},$$

and is zero if $x'(0)=0$; *i.e.*, the initial velocity is zero. Also,

$$\alpha^2-\beta^2-a_1\alpha+a_0$$
$$=\frac{R^2}{4m^2}-\frac{D}{m}+\frac{R^2}{4m^2}-\frac{R^2}{2m^2}-\frac{R}{m}\cdot\frac{x'(0)}{x(0)}+\frac{a}{mx(0)}$$

and putting $x'(0)=0$, we get

$$\frac{a}{mx(0)}-\frac{D}{m}=0$$

or

$$x(0)=\frac{a}{D}. \qquad (4)$$

Hence if the system has an initial displacement $x(0)=a/d$ and a zero initial velocity, then we can have the output response attain its steady-state value instantaneously and with no overshoot.

In order that the system may have an initial velocity $x(0)$, we have to excite the system with another input applied beforehand. Assuming that the required step input for this initial excitation is of amplitude b and is applied to the system at rest (*i.e.*, having no initial velocity or displacement), we get

$$x(t')=\frac{b}{D}+\frac{b/m}{\sqrt{\frac{D}{m}\left(\frac{D}{m}-\frac{R^2}{4m^2}\right)}}\cdot e^{\alpha t}\sin(\beta t+\psi') \qquad (5)$$

where $x(t')$ is the output at a time t' for this initial step b and

$$\psi'=\tan^{-1}\frac{\beta}{-\alpha}.$$

We know that in an underdamped system with a step input, the first zero of velocity occurs at the peak of the overshoot, for which

$$\beta t+\psi'=\pi/2.$$

Calling this time T, we have

$$T=\frac{\pi/2-\psi'}{\beta}. \qquad (6)$$

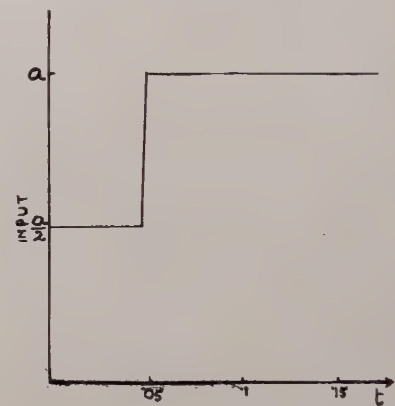
For an underdamped system with a low value of α (of the order of 0.1 or less), we have

$$\psi'\simeq-\pi/2 \quad\text{and}\quad \beta=\omega_0$$

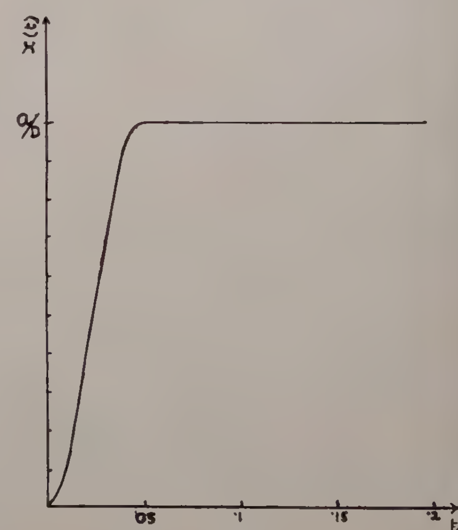
$$\therefore\ T\simeq\frac{\pi}{\omega_0}=\frac{1}{2f_0}.$$

As $T\ll 1$ for any value of $f_0>1$ and as α has already been assumed small,

$$e^{-\alpha T}=1-\alpha T.$$



(a)



(b)

Fig. 1—(a) Modified step input for getting improved response in a system with $f_0=10$ cps and having a low value of effective damping coefficient. (b) Output response of the given system with a modified step input as shown in (a).

* Received by the PGAC, December 30, 1959.

[1] O. J. M. Smith, "Posicast control of damped oscillatory systems," Proc. IRE, vol. 45, pp. 1249–1255; September, 1957.

[2] G. H. Tallman and O. J. M. Smith, "Analog study of dead-beat posicast control," IRE Trans. on Automatic Control, no. PGAC-4, pp. 14–21; March, 1958.

Therefore, in order to have a displacement $x(0)$ with zero velocity, we have, from (5),

$$x(0) = x(T) = \frac{b}{D}(2 - \alpha T) \simeq \frac{2b}{D},$$

for all practical purposes.

But from (4) we must have

$$x(0) = \frac{a}{D}$$

$$\therefore\ b = \frac{a}{2};$$

i.e., the initial step should be half the amplitude of the original step a and should precede it by a time interval T.

Thus the input step signal has to be modified into a two-step staircase signal as shown in Fig. 1(a), with a delay of $1/2f_0$ between the two steps. It is evidently the same type of input modificaion as was developed by Smith from a slightly different approach and was named by him as "Posicast control."[1] The above method of analysis may also be extended to the three-step (first positive, then negative, and then again a positive step) input modification of Smith to get a quicker output response.

Fig. 1(a) and 1(b) show the modified input signal and the corresponding output response for an underdamped system (with $\alpha = 0.1$), having $f_0 = 10$ cps. The immense improvement in output response is evident from Fig. 1(b), which shows a very sharp response with no overshoot. The output reaches its steady-state value *as soon as* the second step is applied and the total time delay between the application of the input signal (the first step) and the steady value of output response is only $1/2f_0$. (A much smaller time delay between the application of the first step and the steady-state output response is possible with a three-step input modification as discussed by Smith[1] in his pioneering work.)

The input modification is independent of the damping coefficient (α) for all practical purposes, provided α is small. But this is not a serious problem, as the effective damping coefficient of any system can be made small by applying a suitable amount of positive feedback. The input modification depends only on the natural resonant frequency (f_0) of the system, which can be determined easily.

Thus, for a given system or for a number of systems having the same natural resonant frequency, the above type of input modification offers a good technique of getting an improved system response. The only disadvantage of the above arrangement is that the input signal has to be remodified for systems having different resonant frequencies.

Input modification based on a similar analytical approach may as well be used profitably for other types of input signals.

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Contributors

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Isaac M. Horowitz (S'52–A'53–M'58–SM'60) was born in Safed, Israel, on December 15, 1920. He came to Canada in 1925, and later attended the University of Manitoba, receiving the B.S. degree in math and physics in 1945. He came to the United States in 1951 and studied at Massachusetts Institute of Technology, Cambridge, Mass., where he received the B.S. degree in electrical engineering in 1952, and at Polytechnic Institute of Brooklyn, where he received the M.S.E.E. degree in 1953 and the D.E.E. degree in 1956. For the next two years he was an assistant professor in the Department of Electrical Engineering at P.I.B.

Since 1958 he has been associated with the Hughes Research Laboratories, Culver City, Calif., in the Exploratory Studies Department. He has done research in magnetic amplifiers, active network synthesis, and feedback theory.

In 1956 he won the National Electronics Conference award for the best paper presented at the Conference.

Mr. Horowitz is a member of AIEE.

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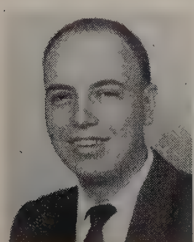

G. W. Johnson

George W. Johnson (A'53) was born on January 12, 1927, in Pawtucket, R. I. He received the B.E.E. degree from Rensselaer Polytechnic Institute, Troy, N. Y., in 1950, and the M.S. degree in electrical engineering from the University of Connecticut in 1954. He served as an instructor in electrical engineering at R.P.I. from 1950 to 1952, and as an instructor and assistant professor of electrical engineering at the University of Connecticut from 1952 to 1956. Since 1956 Mr. Johnson has been working for the International Business Machines military products division in Owego, N. Y., where he is currently a senior engineer assigned to the staff of the division manager of advanced systems research. He has published numerous papers in the field of automatic control.

Mr. Johnson is a member of Sigma Xi and Phi Kappa Phi and is a registered professional engineer.

A. S. Lange

Allen S. Lange was born in Chicago, Ill., on February 17, 1921. He graduated from Lyons Township Junior College, La Grange, Ill., in 1940, with a major in English. In 1949, he received the B.S. degree in aircraft design from the University of Michigan, Ann Arbor, and in 1952, he received the M.S. degree from the same institution.

He began his industrial experience in 1940 with the Electromotive Division of the General Motors Corporation as a draftsman. Following this, he was employed by the Vought-Sikorsky Company, and served as a production liaison engineer. He joined the U. S. Army Air Force in 1943. In 1948, he was employed by the Piasecki Helicopter Company where he made field tests and design studies of control systems. In 1949, he joined the University of Michigan's Willow Run Research Center as a research associate in the Rocket Motor Test Laboratory, and in 1951 he was transferred to the Vibration Studies Group of the Research Center as project engineer. In 1952, he joined Lear, Inc., as a research engineer, to engage in the analysis of autopilots and aircraft instruments.

Later in 1952, he joined the Instrumentation Laboratory of the Massachusetts Institute of Technology, Cambridge, Mass., as a research engineer in the applied mathematics group. In 1955, he became affiliated with the Automation Section of Raytheon Manufacturing Company as a senior engineer engaged in the design of automatic control devices for industrial applications. In 1957, he joined the staff of the Bendix Systems Division as the head of the Guidance and Control Department. In addition to the activities described, Mr. Lange has been active as a consultant in several fields, including mechanical design and packaging for the Lekas Manufacturing Company, and analog computer applications for the GPS Instrument Company.

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William M. Mazer (S'47–A'48–M'58–SM'58) was born on June 25, 1924 in Newark, N. J. He received the B.E.E. degree from the College of the City of New York in 1947. He began graduate work in 1948, receiving the M.E.E. and D.E.E. degrees in 1950 and 1958 from the Polytechnic Institute of Brooklyn, Brooklyn, N. Y., where he was a member of the research staff of the Microwave Research Institute from 1956 through 1957.

W. M. Mazer

From 1950 to 1952 he was a project engineer with the Sperry Gyroscope Company, Great Neck, N. Y., where he designed guided missile automatic control systems. He worked for the Special Devices Center, Office of Naval Research, Sands Point, N. Y., for a year as a senior project engineer. From 1954 to 1956 he was a development engineer with the Federal Telecommunications Laboratory, Nutley, N. J., where he was responsible for the development of an analog computer for a missle guidance system. He was a consultant, engaged in simulator and weapons system analysis, from 1957 to May, 1958, when he joined RCA Defense Electronic Products in New York, N. Y., where he has worked on error correcting codes, PCM-FM system studies, and communication system topology.

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R. B. Northrop

Robert B. Northrop (S'55–M'56) was born in White Plains, N. Y., on January 11, 1935. He received the B.S. degree in electrical engineering in June, 1956, from the Massachusetts Institute of Technology, Cambridge. From 1956 to 1959, he was a member of the University of Connecticut electrical engineering staff as a research assistant on the USAF-IBM sponsored research project there. His work included both research in the field of sampled-data control systems and teaching. He received the M.S. degree in electrical engineering from the University of Connecticut in 1958.

He is currently a research assistant in the Department of Zoology at the University of Connecticut doing research under a U.S.P.H.S. grant in the field of muscle physiology. His principal interests include sampled data systems, the applications of servoanalysis and information theory to the functioning of the central nervous system, and bioelectric instrumentation.

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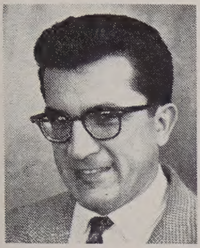

H. A. PAPPO

In 1956 he joined System Development Corporation, Santa Monica, Calif., as a formulator in the air defense system training program for the Semi-Automatic Ground Environment (SAGE) System.

Mr. Pappo is currently engaged in mathematical research in optimization theory in system organization at SDC and is working for the Ph.D. degree in mathematics at the University of California at Los Angeles, specializing in the calculus of variations.

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John J. Rodden was born in San Francisco, Calif. on July 23, 1933. He received the B.S. degree in 1955 and the M.S. degree in 1956 in mechanical engineering from the University of California, Berkeley. He also worked at the University's Engineering Research Institute in heat transfer on boundary layers. His graduate research was on stochastic processes in control systems.

J. J. RODDEN

Since joining the Missiles and Space Division of Lockheed Aircraft Corporation, Sunnyvale, Calif., he has worked on design and analysis of servo control and guidance systems. This has included directing design analysis of a re-entry body attitude control system and the development of a servo-controlled 100 per cent variable thrust liquid rocket engine in conjunction with Naval Ordnance Testing Station at China Lake, Calif. He has worked in the field of human engineering on the analytic design and testing of control systems incorporating human operators.

Mr. Rodden is a member of Tau Beta Pi, Pi Tau Sigma, and Sigma Xi.

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Donald Schulkind (S'53–A'55) was born in New York, N. Y. on February 4, 1932. He attended Columbia University, receiving the B.A. degree in 1953 and the B.S.E.E. degree in 1954. In 1957 he received the M.E.E. degree from Polytechnic Institute of Brooklyn where he is now a candidate for the Doctorate degree in electrical engineering.

D. SCHULKIND

He joined the Sperry Gyroscope Company, Great Neck, N. Y. in 1956 and was initially employed as an associate engineer in the Pod Guidance and Control Engineering Department. He was assigned to his present position as an engineer in the Air Armament Division in 1957, and is currently responsible for the accuracy control of a weapons system. He is also a part-time instructor in the evening session of the Graduate Electrical Engineering Department of the Polytechnic Institute of Brooklyn.

Mr. Schulkind is a member of Tau Beta Pi and Sigma Xi.

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Paul L. Simmons was born on February 12, 1929, in Hutchinson, Kans. He received the B.A. degree in 1953 from the University of Denver with an area major in the humanities and an emphasis on philosophy. After serving two years with the Army, 1953–1955, he received the M.S. degree in library science from the University of Denver in 1956. He then became serials librarian at the Long Beach State College Library, Long Beach, Calif, acting in this capacity from 1956–1958. He joined the System Development Corporation, Santa Monica, Calif., in 1958, where he is presently reference librarian.

P. L. SIMMONS

Mr. Simmons has done bibliographic searches on Soviet aircraft and missiles, mechanical translation, computers in psychology, Soviet computers, tracking, and computers in medicine, in connection with SDC projects on air defense systems and large-scale control systems.

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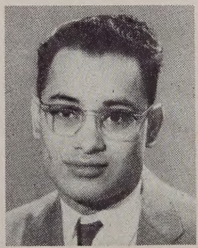
Rangasami Sridhar was born in Bangalore, India, on April 10, 1935. He received the B.E. degree in mechanical engineering in 1955, and the B.E. degree in electrical engineering in 1956, both from the University of Mysore, India. He received the M.S. degree in mechanical engineering in 1957 and the Ph.D. degree in January, 1960, both from Purdue University, Lafayette, Ind. His work for the doctoral dissertation was concerned with signal stabilization of nonlinear feedback systems using external random inputs.



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He is currently an assistant professor of electrical engineering at Purdue University.

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